VIBRATION AND BUCKLING OF GENERALLY ORTHOTROPIC PLATES

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CERTIFIED that this work has been carried out under my supervision and that it has not been submitted else-where for a Degree.

(Dr. V. SUNDARARAJAN)

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NOTATION

x, y, z	Rectangular coordinates
a	Length of plate along x-axis
b	Width of plate along y-axis
h	Thickness of plate along z-axis
u, v, w	Components of displacements along x, y and z directions.
S_1, S_2, S_3	Strains along x, y, and z directions.
T_q $(q = 1, 2 \ldots 6)$	Normal and shear stresses defined in equation A-3.
$b_{qr}(q, r = 1, 2 \ldots 6)$	Elastic stiffness of generally orthotropic plate defined by equation A-2.
M_1, M_2, M_{12}, M_{21}	Bending and Twisting moments per unit length.
N_1, N_2	Inplane loads parallel to x and y directions per unit length.
Q_1, Q_2	Plate shears
D_{ij} $(i, j = 1, 2 \ldots 6)$	Flexural and twisting rigidities defined by equation 2.7
γ	Mass per unit area of the plate = ph
ν	Poissons ratio
p	Inertia force acting on unit area of plate
i	$\sqrt{-1}$
w	Circular frequency
t	Time
C_{mn}	Coefficients in series expansion of plate deflection
θ	Angle of orthotropicity
U	Total potential energy of plate

R_1, R_2, R_{12}	Non-dimensional inplane load coefficients
$D'_{126} = D'_{12} + 2\,D'_{66}$	
Z	Non-dimensional frequency parameter $= (\omega^2 \gamma\, a^4 / \pi^4\, D'_{11})^{1/2}$
R	Side ratio (a/b)
I	Unitary matrix
C	Column matrix of coefficients C_{mn}
ϕ	Beam characteristic function
B_c, L_c	Constants in beam characteristic functions
ℓ	Length of beam
C_{mr}, C_{ns}	Definite integrals equation 2.
D_{mr}, D_{ns}	
E_{mr}, E_{ns}	
F_{mr}, F_{ns}	
λ	Non-dimensional frequency parameter $= (\rho h\, \omega^2 a^4 / \pi^4 D'_{22})^{1/2}$
X_b, X_s	Non-dimensional normal and shear buckling parameters

A B S T R A C T

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Vibration and buckling characteristics of thin, rectangular plates, with arbitrary orientation of orthotropicity are studied. Deflections are assumed to be small and the effects of shear deformation and rotatory inertia are neglected. Approximate solution of the governing differential equation is obtained by the principle of minimum potential energy using a 16 mode Raleigh Ritz procedure. Beam characteristic functions have been used as "admissible functions" to represent the plate deflections. The required integrals of these functions are evaluated and presented. The variational equations so obtained are general in nature and are used to find the non-dimensional frequency and critical buckling load parameters of plates with different combinations of simply supported, clamped and free edges, by solving the correspondin eigenvalue problem. Numerical results of natural frequency and buckling loads are presented for Maple plywood plates having various arbitrary boundary conditions at the edges, different side ratios and angles of orthotropicity. The

results for the specially orthotropic case ($\theta = 0^{\circ}$ or 90°) are compared with previously published results and are found to be in close agreement. The stability and vibration characteristics of a simply supported plate with uniform normal inplane load on a pair of edges is studied. The buckling of simply supported plate subjected to shear loads at the edges is investigated and numerical results are presented for a Mahagony plywood plate.

Graphs are included to show the effect the angle of orthotropicity of the plate on the frequency and critical buckling loads.

CHAPTER-I

INTRODUCTION

1.1 GENERAL

Orthotropic materials play an important role in modern technology. The conflicting demands of increase in strength and stiffness on the one hand and reduction in weight on the other, have led to the use of laminated, stiffened or reinforced construction. Such structural elements are extensively being used in aircraft, missile and ship construction. In the past, materials, regardless of their composition and construction, were generally assumed to be homogeneous and isotropic because of the resulting simplification in the analysis. The present day sophisticated technology, however, requires that the static and dynamic behaviour of orthotropic structures be analysed fully. While certain materials like wood, are orthotropic by nature, a great variety of built up plate-like structures exhibit artificial orthotropicity. Such materials have different elastic properties in different directions. They have, however, three mutually perpendicular planes of elastic symmetry. In an orthotropic plate one of the planes of symmetry is parallel to the plane of the plate. A rectangular plate is "specially orthotropic" if its sides are parallel to the remaining two planes of elastic symmetry, otherwise it is termed as "generally orthotropic". While the isotropic and specially orthotropic plates have received considerable attention of several authors, comparatively less work has been done in the case of

generally orthotropic plate.

1.2 SURVEY OF LITERATURE

a. Vibration Problem:

The transverse vibration characteristics of thin rectangular isotropic plates were analysed and reported in the past. Timoshenko (1)* developed the expression for the potential energy of bending and derived the governing differential equation based on the small deflection thin plate theory. Exact solutions of the differential equation exist when -

i) all edges are simply supported and

ii) a pair of opposite edges simply supported with arbitrary boundary conditions at the other edges (method of Levy).

Warburton (2) determined approximate frequencies of rectangular isotropic plates, subjected to different combinations of free, supported or clamped edges. He applied the Rayleigh method**, representing the deflections by suitable characteristic functions, which satisfy the boundary conditions. This method, however, yields the frequencies which are higher than the exact values. Young (3) used the Raleigh Ritz's method to find the approximate natural frequencies of isotropic rectangular plates by using the beam characteristic functions as "admissible" functions to represent the deflection. Numerical results were given

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*Numbers in the parentheses designate references at the end of the Thesis.

**In this method the frequency is obtained by equating the maximum kinetic energy of the system to its maximum potential energy.

for square plates with

i) all edges clamped,

ii) one edge fixed and remaining free (cantilever)

and iii) two adjacent edges clamped and the remaining free.

Barton (22) extended the treatment of Young to rectangular and skew cantilever plates with various side ratios. He also verified the results experimentally. The experimental values of frequencies were found to be less than those determined from theoretical analysis.

Lekhnitski (24) derived the basic equations of anisotropic elasticity and has considered at length many problems of stress distribution and deformation. Hearmon (5) considered the Hooke's law in its most general form and gave the expressions for the stiffnesses and compliances of orthotropic plates in any arbitrary direction by coordinate transformation. Hearmon and Adams (6) compared the deflection pattern of thin rectangular plates of metal and plywood (cut at various angles to the grain direction) as found from experiments with that from the theoretical analysis, when they are subjected to uniform bending and/or twisting moments at the edges. The results support the theory of bending and twisting of orthotropic plates. Hoppmann and Baltimore (7) proposed that an isotropic plate stiffened by ribs can be treated as a homogeneous orthotropic plate by finding the equivalent elastic constants. Knowing these elastic moduli it is possible to predict the behaviour of the plate subjected to any boundary conditions. The experimental results were in close agreement with the

theoretically predicted values.

Huffington and Hoppmann (4) used the Levy's method to find the natural frequency of rectangular, thin, specially orthotropic plates with a pair of opposite edges simply supported and with arbitrary boundary conditions on the remaining edges. Frequency equations and modal eigen functions were derived. The arbitrary boundary conditions included various combinations of free, supported, clamped or elastically restrained edges. Hearmon (8) used the Raleigh's method to find the approximate frequencies of vibration of specially orthotropic plates under different combinations of clamped or supported edges. Closed form expressions were derived for the frequencies using beam characteristic functions to represent the deflection. Numerical results of the fundamental frequency of square plates were given for six combinations of supported and clamped edges. Somayajulu and Srinivasan (9) extended the method of Huffington and Hoppmann to find the first six frequencies of vibration of specially orthotropic plates with different side ratios. They, further, used the Raleigh Ritz method to determine the first five frequencies of vibration of specially orthotropic cantilever plates. Numerical results of these frequencies were given for five materials with different orthotropic properties and side ratios.

Calligeros and Dugundji (10) investigated the supersonic flutter of generally orthotropic panels using the principle of minimum potential energy and the Raleigh-Ritz method. They have plotted the frequencies of natural vibration

for the first sixteen modes of such panels with three different side ratios and two sets of orthotropic properties.

Weeks and Shidler (11) considered the vibration characteristics of thin, rectangular, specially orthotropic plates, with inplane loads, subjected to different combinations of supported, clamped and elastically restrained edge conditions. The Galerkin's method was used to derive the frequency equations. Dickinson (12) used a sine series solution to analyse the free vibration of specially orthotropic, thin rectangular plates. The method of solution is applicable to plates with any edge conditions expressible using sine series. Application to plates with the following boundary conditions was given :

i) A pair of opposite edges simply supported and each of the remaining edges being simply supported, free, clamped or partially restrained,

ii) all edges clamped and

iii) two opposite edges clamped and the remaining edges free.

Numerical results were given for square plates with (i) all edges clamped and (ii) two opposite edges clamped and the remaining being free. The results were compared with those already available in the literature. The accuracy of the numerical results depends on the convergence of the roots of the determinantal equation.

b. Stability Problem:

The buckling of isotropic plates was discussed at length by Timoshenko and Gere (13). The governing differential

reverse ir

equation and expressions for potential energy were derived. Exact solutions for the critical buckling loads were given for plates with all edges simply supported. The governing differential equation was also solved for plates with the pair of loaded edges simply supported and with arbitrary boundary conditions at the remaining edges, using Levy's method. Maulbetsch (14) used the energy method* to find the approximate critical buckling loads of rectangular isotropic plates with clamped edges. Levy (15) obtained an exact solution of the governing differential equation in terms of infinite series to get the critical buckling loads of isotropic plates with clamped edges. The accuracy of the results depends on the convergence of the series and he estimated that the error is of the order of 0.1%. Green and Hearmon (16) derived the differential equation of bending of thin, rectangular, generally orthotropic plates with uniform inplane loads. Using Fourier series method, the cases of a plate with (i) all edges simply supported and (ii) all edges clamped were solved. Results were also given for a plate with a pair of opposite edges simply supported, the remaining being clamped. Numerical values of critical normal axial and shear buckling loads were evaluated for square and infinitely long generally orthotropic plates with support conditions (i) and (ii). They remark that the results obtained are reasonably accurate for square plates but as the side ratio (a/b) increases the accuracy diminishes.

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*Minimizing the potential energy.

Das (17) used the Levy's method to find the critical buckling loads of thin rectangular specially orthotropic plates with the pair of loaded edges simply supported and the remaining edges having arbitrary boundary conditions. Numerical results were given for different types of plywood plates under the following boundary conditions :

i) all edges simply supported

ii) three edges simply supported and the fourth clamped and

iii) three edges simply supported and the fourth free.

Lure (18) has observed that the vibration as well as the buckling analysis of thin rectangular plates leads to the same Eigenvalue problem under certain boundary conditions. He has suggested a method of finding the frequency of natural vibrations from the critical buckling load parameters. Somayajulu and Srinivasan (9) used this analogy to find the critical buckling loads from the frequency data. Weeks and Shideler (11) calculated the buckling characteristics of specially orthotropic plates by noting the fact that the critical buckling load is the lowest inplane load which makes the frequency of transverse vibration of the plate (subjected to inplane load) vanish.

1.3 STATEMENT OF THE PROBLEM

In the present investigation the vibration and buckling characteristics of thin, rectangular generally orthotropic plates are studied. The material is assumed to be linearly elastic and the analysis is based on the small deflection theory

of thin plates. Effects of shear deformation and rotatory inertia are neglected. A sixteen mode Raleigh Ritz procedure along with the principle of minimum potential energy is used to get an approximate solution of the governing differential equation. Beam characteristic functions, which represent the normal modes of vibration of slender beams, are used as "admissible" functions.

Integrals of these functions are evaluated using a numerical integration scheme. A general formulation of the problem was obtained which could be used to find the frequencies of natural vibration as well as the critical buckling loads (shear and normal) of thin rectangular plates with arbitrary orientation of orthotropicity and any boundary conditions at the edges.

The above procedure is applied to find the non-dimensional frequency and critical buckling load parameters of rectangular generally orthotropic plates having the following edge conditions :

i) all edges simply supported,

ii) all edges clamped,

iii) one edge clamped and the rest free (cantilever plate),

iv) three edges simply supported and the remaining free and

v) a pair of opposite edges simply supported and the rest clamped.

CHAPTER - II

FORMULATION AND SOLUTION

2.1 GENERAL EQUATIONS FOR ORTHOTROPIC PLATES

The governing differential equation and the boundary conditions of a generally orthotropic, thin, rectangular plate are derived by applying the principle of minimum potential energy. The plate (figure No.1) is assumed to have three axes of elastic symmetry, one at right angles to the plane of the plate and the other two lying in its plane, making an angle θ with the sides. The small deflection theory of thin plates is utilised in the analysis which is based on the following assumptions.

1. Thickness of the plate is small when compared to its lateral dimensions.

2. Deflection of the middle surface is small when compared to the thickness of the plate.

3. Rectilinear sections which in the undeformed plate were normal to the middle surface remain rectilinear and normal to the bent middle surface and

4. Normal stress T_3 on planes parallel to the middle surface is small in comparison with the stresses T_1, T_2 and T_6 acting in its plane.

Under these assumptions the displacements are linearly related to the distance z from the middle surface of the plate

and are given by

$$u = -z\frac{\partial w}{\partial x} \; ; \qquad v = -z\frac{\partial w}{\partial y}$$

the strains in the plane of the plate are given by

$$\left.\begin{aligned} S_1 &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \\ S_2 &= \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \\ S_6 &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x\,\partial y} \end{aligned}\right\} \quad \ldots 2.1$$

The stress displacement relations are (from eqn. A.5)

$$\left.\begin{aligned} T_1 &= -z\left[b_{11}\frac{\partial^2 w}{\partial x^2} + b_{12}\frac{\partial^2 w}{\partial y^2} + 2b_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right] \\ T_2 &= -z\left[b_{12}\frac{\partial^2 w}{\partial x^2} + b_{22}\frac{\partial^2 w}{\partial y^2} + 2b_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right] \\ T_6 &= -z\left[b_{16}\frac{\partial^2 w}{\partial x^2} + b_{26}\frac{\partial^2 w}{\partial y^2} + 2b_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right] \end{aligned}\right\} \quad \ldots 2.2$$

The stress components T_4 and T_5 are given by (ref. 8)

$$T_4 = \frac{1}{2}\left(z^2 - \frac{h^2}{4}\right)\left[b_{16}\frac{\partial^3 w}{\partial x^3} + (b_{12}+2b_{66})\frac{\partial^3 w}{\partial x^2\partial y} + 3b_{26}\frac{\partial^3 w}{\partial x\,\partial y^2} + b_{22}\frac{\partial^3 w}{\partial y^3}\right]$$

$$T_5 = \frac{1}{2}\left(z^2 - \frac{h^2}{4}\right)\left[b_{11}\frac{\partial^3 w}{\partial x^3} + 3b_{16}\frac{\partial^3 w}{\partial x^2\,\partial y} + (b_{12}+2b_{66})\frac{\partial^3 w}{\partial x\,\partial y^2} + b_{26}\frac{\partial^3 w}{\partial y^3}\right] \quad \ldots 2.3$$

The bending moments, twisting moments and inplane loads per unit length are given by

$$M_1 = \int_{-h/2}^{h/2} T_1\, z\, dz \; ; \quad M_2 = \int_{-h/2}^{h/2} T_2\, z\, dz \; ; \quad M_{12} = \int_{-h/2}^{h/2} T_6 z\, dz.$$

$$M_1 = \int_{-h/2}^{h/2} T_5 \, dz \text{ and } M_2 = \int_{-h/2}^{h/2} T_4 \, dz \qquad \dots 2.4$$

where h is the thickness of the plate.

Substituting equations 2.2 and integrating we get

$$M_1 = -\left(D_{11} \frac{\partial^2 w}{\partial x^2} + D_{12} \frac{\partial^2 w}{\partial y^2} + 2 D_{16} \frac{\partial^2 w}{\partial x \, \partial y}\right)$$

$$M_2 = -\left(D_{12} \frac{\partial^2 w}{\partial x^2} + D_{22} \frac{\partial^2 w}{\partial y^2} + 2 D_{26} \frac{\partial^2 w}{\partial x \, \partial y}\right) \qquad \dots 2.5$$

$$M_{12} = -\left(D_{16} \frac{\partial^2 w}{\partial x^2} + D_{26} \frac{\partial^2 w}{\partial y^2} + 2 D_{66} \frac{\partial^2 w}{\partial x \, \partial y}\right)$$

and the plate shears are given by (from equation 2.3)

$$Q_1 = -\left[D_{11} \frac{\partial^3 w}{\partial x^3} + 3 D_{16} \frac{\partial^3 w}{\partial x^2 \partial y} + (D_{12} + 2D_{66}) \frac{\partial^3 w}{\partial x \partial y^2} + D_{26} \frac{\partial^3 w}{\partial y^3}\right]$$

$$Q_2 = -\left[D_{16} \frac{\partial^3 w}{\partial x^3} + (D_{12} + 2D_{66}) \frac{\partial^3 w}{\partial x^2 \, \partial y} + 3D_{26} \frac{\partial^3 w}{\partial x \, \partial y^2} + D_{22} \frac{\partial^3 w}{\partial y^3}\right]$$

$$\dots 2.6$$

Where $D_{ij} = b_{ij} \; h^3/_{12}$ $\qquad \dots 2.7$

The above equations for the specially orthotropic case are obtained by putting $b_{16} = b_{26} = 0$ and for the isotropic case by putting $b_{11} = b_{22} = E/(1-\nu^2)$ and $b_{66} = G$.

Substituting equations A.9 in 2.7 the rigidities of a generally orthotropic plate (D_{ij}) with an angle of orthotropicity θ, can be expressed in terms of the rigidities of a specially orthotropic plate by a set of equations obtained by replacing b_{ij}

by D_{ij} and b_{ij}' by D_{ij}' in equations A.7 where D_{ij}' are the rigidities of a specially orthotropic plate.

The total potential energy (U) of a thin, rectangular, plate of uniform thickness h undergoing transverse vibrations is given by (ref. 19).

$$U = \frac{1}{2} \int_{-h/2}^{h/2} \int_0^a \int_0^b (T_1 S_1 + T_2 S_2 + T_3 S_3)\, dxdydz$$

$$- \frac{1}{2} \int_0^a \int_0^b \left[pw - N_1 \left(\frac{\partial w}{\partial x}\right)^2 - N_2 \left(\frac{\partial w}{\partial y}\right)^2 - 2N_{12} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dxdy \quad \ldots 2.8$$

Substituting the stress-strain (A.5) and the strain-displacement (2.1) relations we get

$$U = \frac{1}{2} \int_0^a \int_0^b \left[D_{11} \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\, D_{12} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + D_{22} \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \right.$$

$$+ 4\, D_{66} \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + 4 \frac{\partial^2 w}{\partial x \partial y} \left(D_{16} \frac{\partial^2 w}{\partial x^2} + D_{26} \frac{\partial^2 w}{\partial y^2} \right) - pw$$

$$\left. + N_1 \left(\frac{\partial w}{\partial x}\right)^2 + N_2 \left(\frac{\partial w}{\partial y}\right)^2 + 2N_{12} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dxdy \quad \ldots 2.9$$

where $D_{ij} = \int_{-h/2}^{h/2} b_{ij}\, z^2\, dz = b_{ij}\, h^3/12$

The differential equation and the boundary conditions of the plate can be obtained by applying the "Principle of minimum potential Energy". Taking the variation of equation 2.8 and performing the integration by parts, we get

$$
\begin{aligned}
U = \iint_A \Big[& D_{11}\frac{\partial^4 w}{\partial x^4} + 2\,D_{12}\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} \\
& + 4\,D_{16}\frac{\partial^4 w}{\partial x^3\,\partial y} \\
& + 4\,D_{26}\frac{\partial^4 w}{\partial x\,\partial y^3} - p - N_1\frac{\partial^2 w}{\partial x^2} - N_2\frac{\partial^2 w}{\partial y^2} \\
& - 2\,N_{12}\frac{\partial^2 w}{\partial x\,\partial y}\Big]\,\delta w\,dx\,dy \\
& - \int_0^b M_1\,\delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^a dy - \int_0^a M_2\,\delta\left(\frac{\partial w}{\partial y}\right)\Big|_0^b dx \\
& + \int_0^b \left[Q_1 + \frac{\partial(M_{12})}{\partial y} + N_1\frac{\partial w}{\partial x} + N_{12}\frac{\partial w}{\partial y}\right]\delta w\,dy\,\Big|_0^a \\
& + \int_0^a \left[Q_2 + \frac{\partial\,(M_{12})}{\partial\,x} + N_2\frac{\partial w}{\partial y} + N_{12}\frac{\partial w}{\partial x}\right]\delta w\,\partial x\,\Big|_0^b \\
& - 2\left[M_{12}\;\delta w\right]\Big|_0^a\Big|_0^b = 0
\end{aligned}
$$

.... 2.10

where M_1, M_2, M_{12}, Q_1 and Q_2 have the same meaning as in equations 2.5 and 2.6. Since $\delta w(x,y,t)$ is arbitrary, equation 2.10 will be satisfied if the following conditions hold.

1) $$D_{11}\frac{\partial^4 w}{\partial x^4} + 2\,D_{12}\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} + 4\,D_{16}\frac{\partial^4 w}{\partial x^3\,\partial y} + 4\,D_{26}\frac{\partial^4 w}{\partial x\,\partial y^3} - p - N_1\frac{\partial^2 w}{\partial x^2} - N_2\frac{\partial^2 w}{\partial y^2} - 2\,N_{12}\frac{\partial^2 w}{\partial x\,\partial y} = 0$$

... 2.11

This is the governing differential equation of a vibrating plate subjected to inplane loads N_1, N_2 and N_{12} as shown in

figure 1.

ii)

a) Along edges $x = o, a$

(1) Either $M_1 = 0$ or $\delta\left(\frac{\partial w}{\partial x}\right) = 0$

(2) Either $Q_1 + \frac{\partial(M_{12})}{\partial y} + N_1 \frac{\partial w}{\partial x} + N_{12} \frac{\partial w}{\partial y} = 0$ or $\delta w = 0$

b) Along edges $y = o, b$

(1) Either $M_2 = 0$ or $\delta\left(\frac{\partial w}{\partial y}\right) = 0$

(2) Either $Q_2 + \frac{\partial(M_{12})}{\partial x} + N_2 \frac{\partial w}{\partial y} + N_{12} \frac{\partial w}{\partial x} = 0$ or $w = 0$

c) At corners

Either $M_{12} = 0$ or $\delta w = 0$ ... 2.12

This condition can be phipically interpreted to give the boundary conditions of the plate at the edges.

For a specially orthotropic plate the terms containing D_{16} and D_{26} vanish and the Governing differential equation (2.11) is simplified accordingly.

For the isotropic plate with no inplane loads the equation reduces to the well known form

$$\nabla^4 w = \frac{q}{D}$$

where $D = Eh^3/12(1-\nu^2)$

2.2 FREQUENCY OF TRANSVERSE VIBRATION

For a plate undergoing transverse vibrations without any external loads the governing differential equation is obtained by replacing

$$p \text{ by } -\gamma \frac{\partial^2 w}{\partial t^2} \text{ and substituting } N_1 = N_2 = N_{12} = 0$$

in equation 2.11. It then becomes

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2D_{12} \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + 4 D_{16} \frac{\partial^4 w}{\partial x^3 \partial y}$$

$$+ 4 D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + \gamma \frac{\partial^2 w}{\partial t^2} = 0 \qquad \ldots 2.13$$

This equation is difficult to be solved in most cases and exact solutions are known only for some simple cases. For the case of an isotropic or specially orthotropic plate with all edges simply supported, the equation has been solved (ref. 1 and 19). Exact solution also exists when a pair of opposite edges are simply supported for these plates (Levy's method). In this method the deflection is assumed to be

$$w = f(y) \sin \frac{m \pi x}{a} e^{i \omega t} \qquad \ldots 2.14$$

equation 2.14 when substituted in the partial differential equation will convert it into an ordinary differential equation. The frequency equation can then be derived by solving this equation. Huffington and Hoppmann (4) have used this method for specially orthotropic plates.

For other boundary conditions of isotropic and specially orthotropic plates and any boundary conditions of generally

orthotropic plates it is difficult to solve the differential equation exactly. Under these circumstances we have to take recourse to approximate methods. Raleigh's method was used by Warburton (2) for isotropic plates and by Hoppmann and Baltimore (7) for specially orthotropic plates. This method, however, cannot be applied to the generally orthotropic plates because of the occurence of terms containing D_{16} and D_{26}. For the same reason Galerkin's method also cannot be applied for this case. In the present study the Raleigh Ritz method is used to obtain approximate solution to the governing differential equation. This method was used by several previous authors. Young (3) applied it to isotropic plates, Somayajulu and Srinivasan (9) for specially orthotropic plates and Calligeros and Dugundji (10) to the analysis of orthotropic panel flutter.

2.3 THE RALEIGH RITZ METHOD.

The Raleigh Ritz method, in effect, substitutes a variational problem for the usual characteristic value problem. In a variational problem it is not necessary to impose all boundary conditions in advance, in order to single out a specific solution. The vanishing of the first variation of the functional not only implies the Euler's equations, but also the natural boundary conditions. Courant (23) remarks that for rigid boundary conditions the approximation of the Raleigh Ritz method is good and a few admissible coordinate functions would in most cases, suffice to yield the desired convergence. But the boundary conditions impose a restriction on the choice of functions to represent the deflection. For free boundaries, however, the choice of functions is unlimited, but the convergence is rather slow and it

becomes necessary to take more terms in the series to get the desired accuracy.

In applying the Raleigh Ritz method, the deflection of the plate is assumed as a linear series of admissible coordinate functions. The coefficient of each term of the series is adjusted so as to minimize the potential energy (V) of the plate. The deflection W of the plate can be assumed as

$$W(x,y,t) = \sum_{m=1}^{p} \sum_{n=1}^{q} C_{mn} \; X_m(x) \; Y_n(y) e^{i\omega t} \qquad \ldots 2.15$$

where the functions X_m and Y_n are "admissible" functions i.e. they satisfy the artificial (or rigid) boundary conditions and need not satisfy the natural boundary conditions. In the case of plates prescribed values of slope and deflection constitute the artificial boundary conditions and the values of moments and shear force constitute the natural boundary conditions. Better convergence, however, can be obtained if the natural boundary conditions ~~conditions~~ are also satisfied by the assumed functions.

By substituting equation 2.15 in equation 2.9 the total Potential Energy U can be expressed as a function of the coefficients C_{mn}

$$U = f \; (C_{11}, C_{12} \; --- \; C_{pq})$$

for minimum potential energy we have $\delta u = 0$

$$\text{or} \quad \sum_{m=1}^{p} \sum_{n=1}^{q} \frac{\partial u}{\partial C_{mn}} \delta \; C_{mn} = 0$$

i.e. $\frac{\partial U}{\partial C_{mn}} = 0, (m = 1,2 --- p; n = 1,2 --- q)$ . . . 2.16

Equations 2.16 represent a system of homogeneous, linear equations in the unknown quantities C_{mn}. There are p x q equations in p x q unknowns, which can be determined by equating the determinant of the coefficient matrix to zero. The Eigenvalues of the system can be determined by any numerical technique, using a high speed electronic digital computer.

2.4 PLATE SIMPLY SUPPORTED ON ALL EDGES.

For a plate with all edges simply supported the deflection can be assumed as

$$w(x, y, t) = \sum_{m=1}^{p} \sum_{n=1}^{q} C_{mn} \operatorname{Sin} \frac{m \pi x}{a} \operatorname{Sin} \frac{n \pi y}{b} e^{i \omega t} \quad \ldots 2.17$$

This satisfies the geometric boundary conditions

$$w(x,o,t) = w(x,b,t) = w(o,y,t) = w(a,y,t) = o$$

But, the natural boundary conditions

$$\left[D_{11} \frac{\partial^2 w}{\partial x^2} + 2 D_{16} \frac{\partial^2 w}{\partial x \partial y} + D_{12} \frac{\partial^2 w}{\partial y^2}\right]_{\substack{x = a \\ x = 0}} = 0 \text{ and}$$

$$\left[D_{22} \frac{\partial^2 w}{\partial y^2} + 2 D_{26} \frac{\partial^2 w}{\partial x \partial y} + D_{12} \frac{\partial^2 w}{\partial x^2}\right]_{\substack{y = b \\ y = 0}} = 0$$

are not satisfied.

Substituting 2.17 in 2.9 and noting that for the

vibration problem $p = -\gamma \frac{\partial^2 w}{\partial t^2}$ we get, after simplification,

$$U = \frac{1}{2}\Bigg[\left(\frac{D_{11} m^4 \pi^4}{a^4} + \frac{2 D_{12} m^2 n^2 \pi^4}{a^2 b^2} + \frac{D_{22} n^4 \pi^4}{b^4} - \gamma w^2 \right)$$

$$\int_0^a \int_0^b \left(\sum \sum C_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right)^2 dx\, dy$$

$$+ \frac{4 m^2 n^2 \pi^4 D_{66}}{a^2 b^2} \int_0^a \int_0^b \left(\sum \sum C_{mn} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \right)^2 dx dy$$

$$+ \frac{N_1 m^2 \pi^2}{a^2} \int_0^a \int_0^b \left(\sum \sum C_{mn} \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right)^2 dx\, dy$$

$$+ \frac{N_2 n^2 \pi^2}{b^2} \int_0^a \int_0^b \left(\sum \sum C_{mn} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \right)^2 dx\, dy$$

$$+ \frac{2 N_{12} mn \pi^2}{ab} \int_0^a \int_0^b \left(\sum \sum C_{mn} \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right)$$

$$\left(\sum \sum C_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right) dx\, dy - \left(\frac{4 D_{16} m^3 n \pi^4}{a^3 b} \right.$$

$$\left. + \frac{4 D_{26} m n^3 \pi^4}{a b^3} \right) \int_0^a \int_0^b$$

$$\left(\sum \sum C_{mn} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \right) \left(\sum \sum C_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right)$$

$$dx dy \Bigg] e^{2 i \omega t} \qquad \ldots \qquad 2.18$$

By substituting equation 2.18 in equation 2.16, carrying out the differentiation, simplifying and using the following integrals

$$\int_0^a \sin\frac{m\pi x}{a}\,\sin\frac{n\pi x}{a}\,dx = \int_0^a \cos\frac{m\pi x}{a}\,\cos\frac{n\pi x}{a}$$

$$= 0 \text{ if } m \neq n, \; = \frac{a}{2} \text{ if } m = n$$

and $$\int_0^a \sin\frac{m\pi x}{a}\,\cos\frac{n\pi x}{a}\,dx = 0 \text{ if } m \pm n \text{ is even}$$

$$= \frac{2a\,m}{\pi(m^2 - n^2)} \text{ if } m \pm n \text{ is odd}$$

we get

$$\left[m^4 \frac{D_{11}}{D_{11}'} + 2R^2 \frac{D_{126}}{D_{11}'}(mn)^2 + \frac{R^4 D_{22}\, n^4}{D_{11}'} + R_1 m^2 + R_2 n^2 - Z \right] C_{mn}$$

$$-2 \sum_{R=1}^{p} \sum_{S=1}^{q} C_{rs} \left[\frac{R\, D_{16}(m^2+r^2)}{D'_{11}} + \frac{R^3 D_{26}(n^2+s^2)}{D'_{11}} + R\, R_{12} \right]$$

$$K_{mnrs} = 0$$

$$(m = 1,2 \;\ldots\; p; \quad n = 1,2 \;\ldots\; q) \qquad \ldots\ldots \qquad 2.19$$

where

$D_{126} = D_{12} + 2D_{66}$, $\qquad Z = \omega^2 \gamma\, a^4/\pi^4 D'_{11}$

$R_1 = N_1 a^2/\pi^2 D'_{11}$, $\qquad R_2 = N_2 a^2/\pi^2 D'_{11}$

$R_{12} = N_{12} a^2/\pi^2 D'_{11}$, $\qquad R = a/b$ (side ratio)

and

$$K_{mnrs} = \frac{16\ mnrs}{\pi^2 (r^2-m^2)(s^2-n^2)} \quad \text{when } m \pm r \text{ is odd and } n \pm s \text{ is odd}$$

$$= 0, \text{ when } m \pm r \text{ is even or } n \pm s \text{ is even}$$

The torsional and flexural rigidities D_{ij} for any angle of orthotropicity, θ can be obtained from equations 2.7 and A.9. For the free vibration problem ($R_1 = R_2 = R_{12} = 0$), it is convenient to write the set of equations in the matrix form

$$[K]\{c\} - Z[I]\{c\} = 0 \qquad \dots 2.20$$

$[K]$ will be a diagonal matrix when $\theta = 0$ or 90 and for other angles it will be real, symmetric and positive definite. The Eigenvalues of $[K]$ which will be real and positive represent the natural frequencies of transverse ~~with~~ vibration of the plate. The Jacobi's method is used for solving the eigenvalue problem represented by 2.20. This is a method of diagonalization by successive rotations and iterates to all eigenvectors and eigenvalues simultaneously. This procedure consists of multiplications by matrices, which are similar in form to coordinate transformation matrices that represent angular rotations. The successive multiplications result in the gradual increase of the diagonal terms at the expense of the off-diagonal elements. When finally the off-diagonal elements become zero, the diagonal terms of the resulting matrix are the eigenvalues and the continuous product of rotation is the modal matrix. The method is readily applicable to real and symmetric matrices using the

high speed electronic digital computer.

2.5 PLATE WITH ARBITRARY BOUNDARY CONDITIONS.

For a plate with arbitrary boundary conditions at the edges we can assume

$$w\,(x,y,t) = \sum_{m=1}^{p} \sum_{n=1}^{q} C_{mn}\; X_m\,(x)\, Y_n\,(y)\, e^{i\omega t} \qquad \ldots\ 2.21$$

where X_m and Y_n are both "admissible" functions i.e. they satisfy the rigid boundary conditions. In the present analysis the "beam characteristic functions" are used for these functions. These represent the normal modes of vibration of uniform, long and slender beams. They ~~xx~~ were used as admissible functions for plate vibration problems by several previous authors Warburton (2) and Young (3) used them for isotropic plates and Hearmon (8) and Somayajulu and Srinivasan (9) for specially orthotropic plates. These functions, because of their orthogonal property, are simple to use and many of the Integrals required for the computations for certain boundary conditions are calculated and tabulated by Young (3). The other integrals required for the generally orthotropic plate for all boundary conditions are evaluated and tabulated in the present investigation.

Beam Characteristic Functions:

These functions are obtained from the solution of the differential equation governing the transverse vibration of a uniform beam. The general form of these functions is given by

$$\phi_c = \text{Cosh}\ B_c\,x + E\ \text{Cos}\ B_c x - L_c\,(\text{Sinh}\ B_c x + F\ \text{Sin}\ B_c x) \quad \ldots\ 2.22$$

These are, thus, an infinite number of functions corresponding to $c = 1, 2 \ldots \infty$ representing the different modes of vibration. While the equation 2.22 represents the general form of these functions, for any particular boundary conditions of the beam the corresponding values of E, F, B_c and L_c are to be substituted. These functions and constants are given by Bishop (21), Hearmon (5) and Young (3). The data compiled from these references is given in table 1 for different boundary conditions.

These functions possess the important property of orthogonality i.e. for any two functions ϕ_m and ϕ_n

$$\int_0^{l} \phi_m \phi_n \, dx = l \quad \text{if } m = n$$
$$= 0 \quad \text{if } m \neq n \qquad \ldots 2.23$$

and

$$\int_0^{l} \phi_m^{11} \phi_n^{11} \, dx = \frac{B^4_m}{l^3} \quad \text{if } m = n$$
$$= 0 \quad \text{if } m \neq n \qquad \ldots 2.24$$

where l is the length of the beam

The characteristic functions for a simply supported beam, which are not given in the table, can be taken as

$$\phi_c = \sqrt{2} \ \text{Sin} \ (B_c \, x/l) \qquad \ldots 2.25$$

These functions also satisfy the orthogonal relations 2.23 and 2.24, if we define $B_c = c \pi$ $(c = 1, 2 \ldots \infty)$.

The lowest mode ($c = 1$) in table corresponds to two nodal lines and when dealing with free-supported and free-free plates modes of lower frequencies corresponding to rigid body translation and rotation are possible. These modes are represented by additional characteristic functions proposed by Warburton (2) -

i) Free-supported plate

$$\phi_1 = \sqrt{3}\ (1 - x/\ell) \qquad \ldots \quad 2.26$$

ii) Free-free Plate

$$\phi_1 = 1.0$$

$$\phi_2 = \sqrt{3}\ (1 - 2x/\ell) \qquad \ldots \quad 2.27$$

These functions also satisfy the orthogonal relations represented by equations 2.23 and 2.24 if we define for these lower modes $B_c = 0$ ($c = 1,2$)

Using these functions for $X_m(x)$ and $Y_n(y)$ in equations 2.21 the total potential energy of the vibrating plate with any combination of free, supported or clamped edges can be expressed in terms of the unknown constants C_{mn}. Substituting 2.21in equations 2.9 and 2.16, carrying out the differentiation and simplifying using the following notation

$$C_{mr} = a\int_0^a X_m'' X_r \, dx \qquad C_{ns} = b\int_0^b Y_n'' Y_s \, dy$$

$$D_{mr} = a\int_0^a X_m' X_r' \, dx \qquad D_{ns} = b\int_0^b Y_n' Y_s' \, dy$$

$$E_{mr} = a\int_0^a X_m'' X_r' \, dx \qquad E_{ns} = b\int_0^b Y_n'' Y_s' \, dy$$

$$F_{mr} = a\int_0^a X_m'' X_r \, dx \qquad F_{ns} = b\int_0^b Y_n'' Y_s \, dy$$

... 2.28

$$G_{mnrs} = F_{mr} F_{sn} + F_{rm} F_{ns}$$

... 2.29

$$H_{mnrs} = E_{rm} C_{ns} + E_{mr} C_{sn}$$

$$P_{mnrs} = C_{mr} E_{sn} + C_{rm} E_{ns}$$

get

$$\sum_r \sum_s \Big[D_1 B_m^4 \delta_{mnrs} + D_3 R^2 G_{mnrs} + D_2 B_n^4 R^4 \delta_{mnrs}$$

$$+ 4 D_4 D_{mr} D_{ns} R^2 + 2 D_5 H_{mnrs} R^2 + 2 D_6 P_{mnrs} R^2$$

$$- \lambda^2 \delta_{mnrs} + N_1 a^2 D_{mr} \delta_{ns} / D_{22}' + N_2 R^2 a^2 D_{ns} \delta_{mr} / D_{22}'$$

$$+ N_{12} a^2 (D_{mr} \delta_{ns} + R^2 D_{ns} \delta_{mr}) / D_{22}' \Big] C_{rs} = 0$$

... 2.30

where

$D_1 = D_{11}/D_{22}'$; $D_2 = D_{22}/D_{22}'$; $D_3 = D_{12}/D_{22}'$

$D_4 = D_{66}/D_{22}'$; $D_5 = D_{16}/D_{22}'$; $D_6 = D_{26}/D_{22}'$

$\delta_{mnrs} = 1$ if $m = r$ and $n = s$

$= 0$ if $m \neq r$ or $n = s$

$\delta_{mr} = 1$ if $m = r$

$= 0$ if $m \neq r$

$\delta_{ns} = 1$ if $n = s$

$= 0$ if $n \neq s$

and $\lambda^2 = \rho h \omega^2 a^4 / D_{22}'$

For the free vibration problem ($N_1 = N_2 = N_{12} = 0$) the equations 2.30 can be written in the matrix form

$$[K]\{C\} = \lambda^2 [I]\{C\} \qquad \ldots\ 2.31$$

The matrix $[K]$ is real and symmetric irrespective of the boundary conditions at the edges of the plate. The eigen-value problem represented by the above equation can be solved to yield the frequencies of transverse vibration of the plate (λ) for any boundary conditions at the edges, angle of orthotropicity (θ) and side ratio (R).

2.6 BUCKLING OF GENERALLY ORTHOTROPIC PLATES

The critical values of forces applied in the middle plane of the plate at which the flat form of equilibrium becomes unstable and the plate begins to buckle can be determined by several methods.

1) The plate is assumed to have initially some curvature or lateral load. The inplane forces required to make deflections tend to grow indefinitely are the critical values (Imperfection method).

2) The plate is assumed to buckle slightly under the action of the middle plane forces and the values of these forces are calculated in order to keep the slightly buckled shape in equilibrium (equilibrium method). The differential equation of the surface is obtained in this case by putting p = o in equation 2.11 and

is given by 2.11

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2D_{126} \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + 4 D_{16} \frac{\partial^4 w}{\partial x^3 \partial y}$$

$$+ 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3}$$

$$= N_1 \frac{\partial^2 w}{\partial x^2} + N_2 \frac{\partial^2 w}{\partial y^2} + 2 N_{12} \frac{\partial^2 w}{\partial x \partial y} \qquad \ldots 2.32$$

If only N_1 is acting, the minimum value of N_1 which satisfies the above equation and the boundary conditions of the plate, is the critical buckling load. This method has been used in the literature for the buckling analysis of plates. Green and Hearmon (16) have used this method for 'generally' orthotropic plates employing the Fourier series method to solve the differential equation Das (17) has used this approach to find the critical buckling loads of rectangular, specially orthotropic plates, using the Levy's method to solve equation 2.32.

3) Where a rigorous solution of equation 2.32 is unknown we can calculate the approximate critical buckling loads by equating the strain energy of bending to the work done by the inplane forces (Energy method).

In this method the plate is assumed to undergo small lateral bending consistent with the boundary conditions. If the work done by the inplane forces is smaller than the strain energy of bending for every possible shape of lateral buckling, the flat form of equilibrium is stable. If it is greater than the strain energy of bending the plate is unstable in the flat form

and undergoes buckling. The inplane loads reach critical values when the work done by the inplane forces is equal to the strain energy of bending. Timoshenko and Gere (13) have used this method to find the critical buckling loads for rectangular, isotropic plates with all edges simply supported.

4) The lowest inplane load which makes the frequency of transwise vibration vanish gives the critical buckling load (Kinetic method). This can be obtained from a plot of the frequency of transwise vibration with inplane load versus the magnitude of the inplane load. This method was used by weeks and Shideler (11) to find the critical buckling loads of specially orthotropic plates.

5) By noting the fact that under certain boundary conditions the buckling and vibration problems are similar Boundary value problems, the results of one can be obtained if the results of the other are available. The critical buckling load parameters can be found from the natural frequency parameters by suitable conversion. This analogy can be clearly seen in the case of orthotropic plate with simply supported edges from equation 2.17. For the vibration problem, in the above equations we have $R_1 = R_2 = R_{12} = 0$, whereas for the buckling problem we have $R_2 = R_{12} = Z = 0$. The equation can then be put in the matrix form

$$[K]\{c\} = Z \quad [I]\{C\} \qquad \ldots 2.33$$

Thus both the problems can be reduced to the same Eigen value problem and the Eigenvalues, Z can be interpreted as the

natural frequencies for the vibration problem and as the critical buckling loads for the buckling problem.

Such a similarity can also be noticed in the case of a plate with the pair of loaded edges simply supported. Again we have for the vibration problem $R_1 = R_2 = R_{12} = 0$ and for the buckling problem $R_2 = R_{12} = Z = 0$. It can be seen that $D_{mr} = (m\pi)^2 \delta_{mr}$ and the eigenvalue problem can be put in the matrix notation (equation 2.33). The Eigenvalues can be suitable interpreted as

$$Z = (\rho h \omega^2 a^4 / D^1_{22})^{1/2} \quad \text{for the vibration problem}$$

$$= N_1\, a^2\, m^2\, \pi^2 / D^1_{22} \quad \text{for the buckling problem.}$$

2.7 CRITICAL BUCKLING LOADS.

In the present study the critical buckling load parameters of thin rectangular generally orthotropic plates are found by the equilibrium method. An approximate solution to the differential equation 2.32 is obtained by the principle of minimum potential energy using the Raleigh Ritz procedure. Characteristic beam functions are used as admissible functions to represent the deflection of the plate. The analogy shown in the previous section between the vibration and buckling problems can be used to find the buckling loads from the frequency data when the plate has all edges simply supported. For arbitrary boundary conditions, however, the eigenvalue problem is to be solved separately.

i) UNI-AXIAL BUCKLING

For a plate with arbitrary boundary conditions at the edges subjected to inplane compressive load parallel to x-axis only, we can put $\lambda = N_2 = N_{12} = 0$ in equations 2.30. They can then be put in the matrix form

$$[K]\{C\} - X_b [G]\{C\} = 0 \qquad \ldots 3.34$$

where $X_b = N_1 a^2 / D^1_{22}$

$[K]$ and $[G]$ are real symmetric matrices.

Premultiplying by $[G]^{-1}$, we get

$[A]\{C\} - X_b [I]\{C\} = 0$ where $[A] = [G]^{-1}[K]$

A is a real non-symmetric matrix and its eigenvalues represent the critical buckling loads for different modes of deformation of the plate. Generally, the lowest critical buckling load is of interest and is determined by finding the largest eigenvalue of $[A]^{-1} = [K]^{-1}[G]$

ii) <u>BI-AXIAL BUCKLING.</u>

The minimum compressive load parallel to x-axis to cause buckling of a plate subjected to another inplane load parallel to y-axis (on the other two edges) can be determined by a procedure similar to the one used in (i). Suitable numerical values are substituted for N_2 in equations 2.30 and the eigenvalue problem is solved exactly in the same manner.

iii) SHEAR BUCKLING.

The minimum critical shear loads along the edges of the

plate to cause buckling of ~~the~~ in rectangular generally orthotropic plates can be determined by putting $\lambda = N_1 = N_2 = 0$ in equations 2.30 which can then be put in the form

$$[K]\{C\} = X_s\,[G']\{C\} \qquad \ldots 3.36$$

where $X_s = N_{12}\, a^2 / D^1_{22}$

The matrices $[K]$ and $[G']$ are real and symmetric and the eigenvalue problem can be solved exactly in the same manner as in (i) and (ii).

CHAPTER - III

NUMERICAL RESULTS AND DISCUSSION

3.1 FREQUENCY OF TRANSVERSE VIBRATION.

1) PLATE SIMPLY SUPPORTED AT ALL EDGES: The natural frequencies of transverse vibration of a thin rectangular generally orthotropic plate are calculated by solving the Eigenvalue problem represented by equation 2.20. Numerical results are obtained for a maple plywood plate for which the elastic properties are $D'_{11}/D'_{126} = 1.543$ and $D'_{22}/D'_{126} = 4.813$. The rigidities D_{11}, D_{22}, D_{12}, D_{16} and D_{26} are calculated from equations A.9 and 2.7. The elements of the matrix K are evaluated from equations 2.19 taking p = q = 4 for different side ratios (a/b) and angles of orthotropicity (θ). For the specially orthotropic case (i.e. $\theta = 0°$ or $90°$) K is a 16 x 16 diagonal matrix and for the generally orthotropic plate (for other values of θ) the orthotropic coupling terms containing D_{16} and D_{26} contribute the off-diagonal terms. However, the diagonal terms are large when compared to others and the matrix is real and symmetric. The eigenvalues of K are evaluated by the Jacobi method. The non-dimensional frequency parameter $\lambda = (\rho h \omega^2 a^2 b^2/\pi^4 D_{126})^{1/2}$ is calculated for the first sixteen modes of vibration. These modes are characterised by m and n (in equations 2.19) which represent the number of half waves into which the plate bends along the x and y directions respectively. The frequencies of the first six modes of vibration are given in table 4 for various side ratios and angles of orthotropicity.

The results for the particular cases when $\theta = 0^{\circ}$ and 90° are compared with those obtained from the closed form expression derived by Hearmon (5) using the Raleigh's method in Table 3. This method gives the exact frequency for the simply supported case. The comparison shows that the results of the present investigation agree exactly (upto three decimal places) with those of Hearmon. The Raleigh's method, however, cannot be applied for other angles of orthotropicity because of the terms D_{16} and D_{26} in equation 2.13. The effect of the side ratio on the frequency of the two modes ($m = 1$, $n = 1$ and $m = 1$, $n = 2$) is shown in figure 2. The frequency decreases rapidly with the increase of the side ratio from a high value, attains a minimum at a particular a/b ratio and increases at a less rapid rate. The value of the side ratio at which the frequency is minimum depends upon the mode shape the angle of orthotropicity and the elastic properties of the plate. Figure 3 shows the variation of the frequency of the first three modes ($m = 1$, $n = 1$, $m = 1$, $n = 2$ and $m = 2$, $n = 1$) with the angle of orthotropicity for side ratios of 0.4, 1.0, 2.0 and 3.0. For certain modes the frequency increases with the angle of orthotropicity while for others it decreases. There are some modes for which the maximum frequency occurs at a value near about 45°.

The frequency of transverse vibration of a plate subjected to normal inplane load along a pair of opposite edges is calculated from equation 2.20. Numerical values (both tensile and compressive) are substituted for N_1 in equations 2.19 keeping $N_2 = N_{12} = 0$ in them to evaluate the elements of the matrix

K, which is real and symmetric. The eigenvalue problem is solved and the frequency parameters are determined and given table 5 for various side ratios, angles of orthotropicity and edge loads. Figure 4 shows the effect of the inplane load on the frequency of transverse vibration. The square of frequency decreases linearly with the decrease of the tensile load (or increase of compressive load) and the inplane load corresponding to zero frequency is the static buckling loads of the plate.

ii) PLATE WITH ARBITRARY BOUNDARY CONDITIONS.

The frequencies of transverse vibration of thin rectangular plate with arbitrary boundary conditions at the edges are found by solving the eigenvalue problem represented by equation 2.31. The numerical values of the integrals required in equation are determined by the integration of the beam characteristic functions. These characteristic functions which represent the mode shapes of long, slender beams subjected to transverse vibrations for different boundary conditions are given in table 1.C$_n$ represents the mode number (the number of half waves into which the beam bends). A numerical integration procedure using the "Romberg Integration" is utilized to evaluate all the required integrals. The numerical results are presented in table 2 (a, b and c). The integrals $\int \phi^1_m \, \phi_n \, dx$ and $\int \phi_m^{11} \, \phi^1_n \, dx$ were not previously reported in the literature whereas the integrals $\int \phi^1_n \, \phi^1_n \, dx$ and $\int \phi^{11}_n \, \phi_n \, dx$ were tabulated by Young (3) for clamped- clamped, clamped - free and

free-free beams. The results given in Table 2, for these cases are in close agreement with those of reference (3). This numerical data is now substituted in equations 2.30, to evaluate the elements of the matrix K. This matrix is real and symmetric irrespective of the boundary conditions of the plate. The eigenvalues are determined as in case (i) and the non-dimensional frequency parameters ($\lambda^2 = \rho h \omega^2 a^4/D^1_{22}$) are evaluated for the first 16 modes of vibration. A general computer program is made (given in Appendix B) to find the frequencies of vibration of thin rectangular plates with any arbitrary boundary conditions at the edges, with side ratios ranging from 0.5 to 3.5 (in steps of 0.5) and with the angle of orthotropicity ranging from 0° to 90° (in steps of 15°).

(Continued....)

Numerical Values of these frequencies for a maple plywood plate ($D'_{11} / D'_{22} = 3.12$, $D'_{12} / D'_{22} = 0.1206$ and $D'_{66}/D'_{22} = .2637$) are evaluated and presented in tables 6a - 6e for the following boundary conditions;

a) all edges simply supported

b) all edges clamped

c) two opposite edges clamped and the others simply supported

d) one edge clamped and the rest free (canti lever plate)

and e) one edge free and the rest simply supported.

While the frequencies of generally orthotropic plates with arbitrary boundary conditions are calculated for the first time, the specially orthotropic case was solved by Hearmon (8) using the Raleigh's method for their rectangular plates having various combinations of clamped and supported edges. The fundamental frequencies found from equation 2.31 are compared with those determined from the closed form expressions derived by Hearmon for specially orthotropic case i.e., when $\theta = 0^{\circ}$ and $\theta = 90^{\circ}$.

The Raleigh's method is known to give exact frequency for the case of a plate simply supported at all edges, whereas, for other boundary conditions it gives an upper bound for the frequencies. Comparison in table 7 shows a very good convergence of the Raleigh Ritz method to these true values for the case plate from the higher side (the difference being of the order of 0.05%). For plates with other boundary conditions, for which comparison could be made, it can be seen that the frequencies found in the

present investigation are less than (of the order of 0.2%) those given by Raleigh's method. This indicates that the Ritz's modification of the Raleigh's method improves the convergence to the true value from the higher side considerably. In table 8 the first five frequencies of cantilever plate obtained from equation are compared with those given by Somayajulu and Srinivasan (9) for the case when $\theta = 0$. The values in reference 9 appear to have been wrongly tabulated - the fundamental frequency for side ratios 0.5 and 2.0 having been omitted. Comparison is made after the necessary correction and values agree very closely. This is to be expected since the equations 2.30 become exactly same as those used in reference 9 for angles of orthotropicity of 0 or 90°. The a/b ratio has practically no effect on the fundamental frequency of vibration of a specially orthotropic cantilever plate. Barton (22) reports a similar behaviour of a rectangular isotropic plate.

The fundamental frequency of vibration is plotted against the angle of orthotropicity in figure 5 and 6 for side ratios (a/b) 0.5, 1.0, 2.0 and 3.0 and various boundary conditions. The frequency of clamped - clamped plate reduces with the increase of the angle of orthotropicity, θ for a side ratio of 0.5 whereas for other ratios it decreases. The cccc plate has the highest frequency and the cantilever plate the lowest. The frequencies of cccc and sscc are wide apart for low a/b ratios and they come closer as this ratio is increased.

3.2 CRITICAL BUCKLING LOADS.

i) PLATE SIMPLY SUPPORTED ON ALL EDGES:

The critical buckling loads of generally orthotropic plates, when all edges are simply supported, can be determined from equation 2.33 by solving the corresponding eigenvalue problem.

a) Uniaxial buckling: For buckling under uniform compressive inplane loads on a pair of opposite edges the eigenvalue problem is exactly same in 3.1 (i). The elements of the matrix k are found from equations 2.19 by putting $N_2 = N_{12} = 0$. The matrix K is found to be real and symmetric and the eigenvalues are determined using the Jacobi's method. The non-dimensional critical buckling load parameters ($K_c = b^2 N_1 / h^3 E_L$) are evaluated for mahogany plywood plate for which $D'_{11}/D'_{22} = 3.04$, $D'_{126}/D'_{22} = 0.438$ and $E_L = 1.35 \times 10^6$ psi. These values for the first three modes of deformation of the plate for various angles of orthotropicity and side ratios are given in table 9. Green and Hearmon (16) have, by using a Fourier Series solution, found the critical buckling loads of generally orthotropic plates and have presented numerical results for a square mahogany plywood plate. They have made approximations by limiting the number of terms in the series to six and simplifying the results to derive closed form expressions for the buckling loads. It is remarked that the results are reasonably accurate for a side ratio of 1 and the accuracy decreases with the increase of the side ratio and the magnitude of the orthotropic coupling terms

(terms containing D_{16} or D_{26}). The present investigation takes 16 term series for the deflection function and is expected to give better results. These results are compared with those of Green and Hearmon in table 9 for a square plate and are found to be in close agreement. In figure 7 the minimum critical buckling load is plotted against the a/b ratio for various angles of orthotropicity. The behaviour is similar to that of specially orthotropic plates as reported by Das (17). For low values of side ratios the first mode gives the minimum critical buckling load and as a/b ratio increases beyond certain value (indicated by a Kink in the curve) the second mode give the minimum buckling load. The angle of orthotropicity alters the minimum buckling loads and the location of the kinks along the curve. For certain a/b ratios the advantage in having a value of θ other than 0° or 90° for increasing the critical buckling load is clearly seen in figure 7.

b) Bi-axial Buckling: The minimum critical normal load parallel to y-axis on two opposite edges required to cause buckling of a thin rectangular generally orthotropic plate, when it is subjected to uniform normal inplane load parallel to the x-axis on the remaining two edges, is determined from equation 2.20. The elements of the matrix K are determined from equations 2.19 by giving suitable values to R_2 keeping $S = R_{12} = 0$. The matrix K is real and symmetric and the eigenvalues are determined by the Jacobi's method. The numerica

results for mahogony plywood plate ($D_{11}' / D_{22}' = 3.04$ and $D_{126}' / D_{22} = 0.438$) are given in table 9. The critical buckling loads for the first three modes of deformation are plotted against the inplane load parallel to the y-axis in figure 4. The buckling load (along y-direction) is found to vary linearly with the uniform inplane load acting parallel to the y-axis. A plate loaded in tension requires a higher compressive load to cause buckling of the plate than that required by an unloaded plate. A compressive load, on the other hand, requires a smaller load to buckle the plate. Timoshenko (13) describes a similar behaviour of isotropic plates under uniform edge loads in two perpendicular directions. The lines in figure 4 corresponding to the modes $m = 1$, $n = 2$ and $m = 2$, $n = 1$ are not symmetrical with reference to the R_1 and R_2 axes whereas for the isometric case such a symmetry exists for these and other similar modes.

c) Buckling Under Shear Loads: The minimum shear loads along the edges of a thin rectangular plate required to cause buckling can be determined from equations 2.19 by putting $Z = R_1 = R_2 = 0$. The lowest value of R_{12} which makes the determinant of the coefficient matrix of equations 2.19 equal to zero is the minimum critical buckling load. Starting from a value of zero for R_{12}, it is increased gradually in small steps and the sign of the determinant is found, when the sign changes the step size is reduced and the process is repeated until a value of R_{12} is found to sufficient accuracy, which makes the determinant zero very nearly. This value is the shear buckling

ətric

load. The numerical results obtained for the mahogony plywood plate are shown in table 11. The results are compared in table 10 with those of Green and Hearmon (16) for the case when a/b = 1.0 and are found to be in close agreement. Figure 9 shows the variation of the shear buckling loads (+ve and -ve) with the angle of orthotropicity.

ii) PLATE WITH ARBITRARY BOUNDARY CONDITIONS.

For a plate subjected to arbitrary boundary conditions at the edges the minimum critical buckling load is found by solving the eigenvalue problem represented by equation 3.34. This is a general form of the eigenvalue problem. The ~~mx~~ minimum eigenvalue of the matrix $[A] = [K]^{-1} \; [G]$ gives the minimum buckling load. This is determined by taking the inverse of the largest eigenvalue of the inverted matrix ($[A]^{-1} = [G]^{-1}[$ A general compurter program is made to evaluate the elements of matrices K , G and A^{-1} and to determine the largest eigenvalue of A^{-1} by an iterative procedure. The numerical values of the minimum buckling loads found by this procedure are given in Table 12 for various boundary conditions.

The variation of the minimum buckling load with the angle of orthotropicity is plotted in figure 9.

3.3 CONCLUSIONS.

The frequency of transverse vibration and the critical buckling loads of thin rectangular generally orthotropic plates were evaluated using the Raleigh Ritz method and the principle of

minumum potential energy. The deflections were assumed to be small and the effects of rotating inertia and shear deformation were neglected. In the vibration problem the frequencies are the eigenvalues of a real symmetric matrix. The natural frequencies of vibration for the first sixteen modes were calculated by using the Jacobi's method on the high speed electronic computer for various boundary conditions. But in the buckling problem the critical buckling loads were the eigenvalues of a real non-symmetric matrix (except when the pair of loaded edges are simply supported, in which case it is real and symmetric). The lowest critical buckling load is determined by an in iterative scheme.

The effect of the angle of orthotropicity on the vibration and critical buckling loads is shown in the several tables and graphs presented. In some cases an orientation of orthotropicity other than 0° or 90° increases the minimum critical buckling load by as high as 100%. The advantage gained however will depend upon the side ratio, the boundary conditions at the edges and the orthotropic properties of the material. In applications where minimum weight is an important design consideration an investigation of this type is worthwhile in view of the improved buckling characteristics at certain angles of orthotropicity.

3.4 SCOPE OF FURTHER WORK.

The accuracy of the results of higher modes of vibration

could be improved by taking into account the effect of shear deformation and rotatory inertia. The analysis could then be applied to thick plates also. Since plate like structural elements, of shapes other than rectangular, are frequently used in air craft and missile applications, it is profitable to extend the present analysis to such shapes (skew, trangular etc.). The effects of large deformations on the vibration characteristic could also be studied for the generally orthotropic plates. Further, the method of the present work could be extended to cylindrical and spherical shells to investigate the effects of angular orthotropicity on the vibration and buckling characteristics.

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24.

APPENDIX 'A'

ELASTIC CONSTANTS FOR ORTHOTROPIC THIN PLATE

A.1 THE GENERALISED HOOKE'S LAW:

Robert Hooke in 1678 proposed his famous law, which in tensor notation can be expressed as

$$T_{ij} = c_{ijk\ell} \; S_{k\ell}$$

$$\text{and} \quad S_{ij} = s_{ijkl} \; T_{k\ell} \; -- (i,\ j,\ k,\ l = 1,\ 2,\ 3) \quad \text{--- A.1}$$

T_{ij} and S_{kl} are components of stress and strain and $c_{ijk\ell}$ and s_{ijkl} are the elastic stiffnesses and compliances respectively. There are, in general, 81 constants in each of equations A.1 but because of the symmetry relations existing between shear stresses and strains ($T_{ij} = T_{ji}$ and $S_{ij} = S_{ji}$), this number reduces to 36. Hearmon (6) shows by a thermodynamic argument that $c_{ijkl} = c_{klij}$ and $s_{ijkl} = s_{klij}$. Thus the number of independent constants in each of the above equations reduces to 21 and the Generalised Hooke's law can be written as

$$\left.\begin{aligned} T_q &= b_{qr} \, S_r \\ \text{and } S_q &= s_{qr} \, T_r \; -(q,\ r = 1,2 \ldots 6) \end{aligned}\right| \quad \ldots \quad \text{A.2}$$

The first equation in the expanded form becomes

$$T_1 = b_{11}S_1 + b_{12}S_2 + b_{13}S_3 + b_{14}S_4 + b_{15}\,S_5 + b_{16}\,S_6$$

$$T_2 = b_{21}S_1 + b_{22}S_2 + b_{23}S_3 + b_{24}S_4 + b_{25}S_5 + b_{26}\,S_6$$

$$T_3 = b_{31}S_1 + b_{32}S_2 + b_{33}S_3 + b_{34}S_4 + b_{35}S_5 + b_{36}S_6$$

$$T_4 = b_{41}S_1 + b_{42}S_2 + b_{43}S_3 + b_{44}S_4 + b_{45}S_5 + b_{46}S_6$$

$$T_5 = b_{51}S_1 + b_{52}S_2 + b_{53}S_3 + b_{54}S_4 + b_{55}S_5 + b_{56}\ S_6$$

$$T_6 = b_{61}S_1 + b_{62}S_2 + b_{63}S_3 + b_{64}S_4 + b_{65}S_5 + b_{66}\ S_6 \qquad A\cdot 3$$

There are thus 21 independent constants in equations A-3, because of the relation $b_{ij} = b_{ji}$. The stiffnesses and compliances relate the components of one second order tensor to those of another (equation A-1) and therefore form a fourth order tensor. They transform from one set of coordinate axes to another according the laws of coordinate transformation. Thus there are 81 transformation equations in 81 constants in the most general case which are reduced to 21 equations in 21 constants due to the relations cited above (5).

A.2 THIN ORTHOTROPIC PLATE.

If a plate is orthotropic, i.e. it possesses three mutually perpendicular planes of elastic symmetry, and one such plane coincides with the plane of the plate, Sokolnikoff (20) has shown that the number of independent elastic constants reduces to 9 and the stress-strain relations become

$$T_1 = b_{11}S_1 + b_{12}S_2 + b_{13}S_3 + 0 + 0 + b_{16}S_6$$

$$T_2 = b_{12}S_1 + b_{22}S_2 + b_{23}S_3 + 0 + 0 + b_{26}S_6 \qquad A.4$$

$$T_3 = b_{13}S_1 + b_{23}S_2 + b_{33}S_3 + 0 + 0 + b_{36}S_6$$

$$T_4 = 0 + 0 + 0 + b_{44}S_4 + b_{45}S_5 + 0$$

$$T_5 = 0 + 0 + 0 + b_{54}S_4 + b_{55}S_5 + 0$$

$$T_6 = b_{16}S_1 + b_{26}S_2 + b_{36}S_3 + 0 + 0 + b_{66}S_6$$

In the case of thin plate, i.e. a plate whose lateral dimensions are large incomparison to its thickness, the state of stress is approximately plane and it can be assumed that $T_3 = T_4 = T_5 = 0$. Such a plate is known as a "generally orthotropic" plate and the Hooke's law in its case can be expressed as

$$T_1 = b_{11}S_1 + b_{12}S_2 + b_{16}S_6$$

$$T_2 = b_{12}S_1 + b_{22}S_2 + b_{26}S_6 \qquad \text{A.5}$$

$$T_6 = b_{16}S_1 + b_{26}S_2 + b_{66}S_6$$

If the two axes of elastic symmetry lying in the plane of the plate are parallel to the x,y coordinate axes, then the plate is known as "specially orthotropic". Then $b_{16} = b_{26} = 0$ and the Hooke's law becomes

$$T_1 = b_{11}S_1 + b_{12}S_2$$

$$T_2 = b_{12}S_1 + b_{22}S_2 \qquad \text{A.6}$$

$$T_6 = b_{66}S_6$$

It can easily seen that

$b_{11} = E_1/\mu$; $b_{12} = E_1 \nu_{21}/\mu = E_2 \nu_{12}/\mu$

$b_{22} = E_2/\mu$, $b_{66} = G_{12}$ where $\mu = 1 - \nu_{12}\nu_{21}$ ... A.7

If the plate is isotropic we have $E_1 = E_2 = E$, $\nu_{12} = \nu_{21} = \nu$ and $\mu = 1 - \nu^2$, then

$$b_{11} = b_{22} = \frac{E}{(1-\nu^2)} , \quad b_{12} = \frac{\nu E}{1-\nu^2}$$

and $b_{66} = G$ ... A.8

the number of independent constants are then only two since a relation exists between E and G.

A.3 COORDINATE TRANSFORMATION.

For an orthotropic thin plate the elastic constants in any arbitrary directions inclined at an angle θ to the axes of elastic symmetry in the plane of the plate are given by coordinate transformation (5)

$$b_{11} = b'_{11}\, m^4 + b'_{22}\, n^4 + 2(b'_{12} + 2\, b'_{33})\, m^2 n^2$$

$$b_{22} = b'_{11}\, n^4 + b'_{22}\, m^4 + 2(b'_{12} + 2\, b'_{33})\, m^2 n^2$$

$$b_{12} = (b'_{11} + b'_{22} - 4\, b'_{66})\, m^2 n^2 + b'_{12}\,(m^4 + n^4)$$

$$b_{16} = \left[b'_{22}\, n^2 - b'_{11} m^2 + (b'_{12} + 2b'_{33})\,(m^2 - n^2) \right] mn$$

$$b_{26} = \left[b'_{22}\, m^2 - b'_{11} n^2 - (b'_{12} + 2b'_{33})\,(m^2 - n^2) \right] mn$$

$$b_{66} = (b'_{11} + b'_{22} - 2b'_{12})\, m^2 n^2 + b'_{66}\,(m^2 - n^2)^2 \qquad \text{A.9}$$

where b_{11}, b_{22} etc. are elastic constants along arbitrary directions b'_{11}, b'_{22} etc. are elastic constants along axes of elastic symmetry $m = \text{Cos}\,\theta$ and $n = \text{Sin}\,\theta$.

APPENDIX 'B'

COMPUTER PROGRAM

```
C     FREQUENCIES OF GENERALLY ORTHOTROPIC PLATES
C     THIS PROGRAM CALCULATES THE ELEMENTS OF THE 16 x 16 MATRIX
C     K, FOR DIFFERENT SIDE RATIOS AND ANGLES OF ORTHOTROPICITY.
C     THE EIGENVALUES ARE DETERMINED BY JACOBI'S METHOD AND THE
C     NON-DIMENSIONAL FREQUENCY PARAMETERS ARE PRINTED FOR THE
C     FIRST 16 MODES OF VIBRATION AS OUT PUT.
      DIMENSION FA(4,4), FB(4,4), FC(4,4), FD(4,4), FE(4,4),FF(4,4)
      DIMENSION BUCK (4,4),A(4,4), D(16,16), EGEN(16), ALAM(16)
     DIMENSION LFO F2 (4,4), LF1F2 (4,4), LF1FO(4,4), Z(16,16),B(6)
      DIMENSION x (6), LF1F1 (4,4), LF2F1 (4,4), LF2FO( 4,4), LFOF1 (4,4)
      BF1FO(4,4), BF1F1(4,4), BF2F1(4,4), BF2FO(4,4), BF1F2( 4,4)
                              BFOF1 (4,4)
      BFOF2 (4,4)
      REAL LFOF2, LF1F2, LF1FO, LF1F1, LF2F1, LF2FO, LFOF1
C     READ INTEGRELS OF BEAM CHARACTERISTIC FUNCTIONS AS DATA
      READ 333, (LF1F1 (M, N), N=1,4), M= 1,4)
      READ 333, (LF1FO (M, N), N=1,4), M= 1,4)
      READ 333, (LF2F1(M,N) , N= 1,4), M = 1,4)
      READ 333, (LF2FO(M,N), N=1,4), M=1,4)
      READ 333, (BF1FO(M,N), N=1,4), M= 1,4)
      READ 333, (BF2F1(M,N), N=1,4) M = 1,4)
      READ 333, (BF2FO(M,N), N=1,4) M= 1,4)
      READ 333, (BF1F1(M,N), N=1,4) M= 1,4)
      READ 334, (X(M), M = 1,4)
      READ 334, (B(M), M= 1,4)
C     PRINT DATA FOR OUTPUT RECORD
      PRINT 333, (LF1F1 (M,N), N= 1,4), M= 1,4)
      PRINT 333, (LF1FO (M,N), N= 1,4), M=1,4)
      PRINT 333, (BF1FO(M,N), N=1,4), M=1,4)
```

```
      PRINT 333, (LF2F0(M,N), N=1,4), M=1,4)
      PRINT 333, (BF1F0(M,N), N=1,4), M=1,4)
      PRINT 333, (BF2F1(M,N), N=1,4, M= 1,4)
      PRINT 333, (BF2F0(M,N), N=1,4), M= 1,4)
      PRINT 333, (BF1F1(M,N), N=1,4) M=1,4)
      PRINT 334 (B(M), M=1,4)
      FORMAT (6F12.3)
33    FORMAT (4F16.8)
34    FORMAT (2 X, 16F8.3)
22    PI=4.*ATAN(1.)
      DO 111 M= 1,4
      DO 111 N= 1,4
      LF0F2 (M,N)= BF2F0(N,M)
      LF1F2(M,N)=LF2F1(N,M)
      BF1F2(M,N)= BF2F1(N,M)
      LF0F1(M,N)= LF1F0(N,M)
      BF0F1(M,N)= BF1F0(N,M)
      B11, B22 ARE ORTHOTROPIC PROPERTIES OF THE MAPLE 5 PLY-
      PLYWOOD PLATE (DATA)
      B11=3.12
      B22=1.00
      B12= 0.1206
      B13=0.0
      B23=0.0
      B33=0.2637
      THETA= -15.0
31    THETA=THETA+15.0
```

```
      T =THETA*PI/180.
      C=COS(T)
      S=SIN(T)
      C2 = C**2
      S2 = S**2
      C4 = C**4
      S4 = S**4
      BXY = B12+2.*B33
C     D11, D22  ARE ORTHOTROPIC PROPERTIES OF THE PLATE FOR
C               ANGLE OF ORTHOTROPICITY THETA
     D11=B11*C4 + B22*S4 +2.*BXY*C2*S2
      D22 = B22*C4 +B11*S4 +2.*BXY*C2*S2
      D13 = (B22*S2-B11*C2+ BXY*(C2-S2))*C*S
      D23 = (B22*C2-B11*S2-BXY*(C2-S2))*C*S
      D12 = (B11 + B22-4.*B33)*C2*S2 + B12*(C4 -S4)
      D33 = (B11 + B22 - 2.*B12)*C2*S2 + B33*(C2-S2)**2
3     R IS SIDE RATIO - A/B
      R = 0.0
4     R = R+0.5
      DO 1 M = 1,4
      DO 1 N= 1,4
      DO 16 J = 1,4
      DO 16 I = 1,4
      BM = B(M)
      XN = X(N)
      BN = B(N)
      IF (M.EQ.J) GO TO 11
      DELMI = 0.0
```

```
      T =THETA*PI/180.
      C=COS(T)
      S=SIN(T)
      C2 = C**2
      S2 = S**2
      C4 = C**4
      S4 = S**4
      BXY = B12+2.*B33
C     D11, D22  ARE ORTHOTROPIC PROPERTIES OF THE PLATE FOR
C               ANGLE OF ORTHOTROPICITY THETA
     D11=B11*C4 + B22*S4 +2.*BXY*C2*S2
      D22 = B22*C4 +B11*S4 +2.*BXY*C2*S2
      D16 = (B22*S2-B11*C2+ BXY*(C2-S2))*C*S
      D26 = (B22*C2-B11*S2-BXY*(C2-S2))*C*S
      D12 = (B11 + B22-4.*B33)*C2*S2 + B12*(C4 -S4)
      D66 = (B11 + B22 - 2.*B12)*C2*S2 + B33*(C2-S2)**2
C     R IS SIDE RATIO - A/B
      R = 0.0
4     R = R+0.5
      DO 1 M = 1,4
      DO 1 N= 1,4
      DO 16 J = 1,4
      DO 16 I = 1,4
      BM = B(M)
      XN = X(N)
      BN = B(N)
      IF (M.EQ.J) GO TO 11
      DSINMI = 0.0
```

```
      T =THETA*PI/180.
      C=COS(T)
      S=SIN(T)
      C2 = C**2
      S2 = S**2
      C4 = C**4
      S4 = S**4
      BXY = B12+2.*B33
C     D11, D22  ARE ORTHOTROPIC PROPERTIES OF THE PLATE FOR
C               ANGLE OF ORTHOTROPICITY THETA
      D11=B11*C4 + B22*S4 +2.*BXY*C2*S2
      D22 = B22*C4 +B11*S4 +2.*BXY*C2*S2
      D13 = (B22*S2-B11*C2+ BXY*(C2-S2))*C*S
      D23 = (B22*C2-B11*S2-BXY*(C2-S2))*C*S
      D12 = (B11 + B22-4.*B33)*C2*S2 + B12*(C4 -S4)
      D33 = (B11 + B22 - 2.*B12)*C2*S2 + B33*(C2-S2)**2
C     R IS SIDE RATIO - A/B
      R = 0.0
4     R = R+0.5
      DO 1 M = 1,4
      DO 1 N= 1,4
      DO 16 J = 1,4
      DO 16 I = 1,4
      BM = B(M)
      XN = X(N)
      BN = B(N)
      IF (M.EQ.J) GO TO 11
      DELMJ = 0.0
```

```
      GO TO 12
10    DELMI = 1.0
12    IF(N.EQ.J) GO TO 11
      DELNJ = 0.0
      GO TO 13
11    DELNJ = 1.0
13    DIMNLJ = DELMI*DELNJ
      CMI = LFIFO(M,I)
      CIM = LFOF1(M,I)
      DMI = LF1F1 (M, I)
      EMI = LF2F1(M, I)
      FMI = LF2FO(M,I)
      EIM = LF1F2(M,I)
      DIM = DMI
      FIM = LFOF2 (M,I)
      CNJ = BF1FO(N,J)
      DNJ = BF1F1 (N,J)
      ENJ = BF2F1 (N,J)
      FNJ = BF2FO(N,J)
      EJN = BF1F2(N, J)
      CJN = BFOF1 (N,J)
      DJN = DNJ
      FJN = BFOF2 (N, J)
      FA(I, J) = D11*XM**4*DIMNLJ
      FB(I,J) = D12*(FIM*FNJ+FMI*FJNI)*R*R
      FC(I,J) = D22*XN**4*R**4*DIMNLJ
      FD(I,J) = 4.*D33*DIM*DMN*DJN*R*R
```

```
      FE(I,J) = 2.*D13*(EMI*CJN+ EIM*CMJ)*R*R
      FE (I,J) = 2.*D23*(CIM*ENJ+ CMI*EJN)*R*R
      ABCD = FA (I,J) + FB(I,J) +FC(I,J) + FD(I,J) +FE(I,J)+ FF(I,J)
      A(I,J) = ABCD
15    CONTINUE
      L = 4*M-4+N
      J4= J+4
      J8 = J+8
      J12 = J + 12
C     D'S ARE THE ELEMENTS OF THE MATRIX K
      D(L,J) = A(1,J)
      D(L,J4) =A(2,J)
      D(L,J8) =A(3,J)
      D(L,J12) = A(4,J)
16    CONTINUE
1     CONTINUE
      PRINT 2,R,RX, THETA
2     FORMAT (2X, 3F20.2)
21    FORMAT (2X, 16F8.1)
C     ELEMENTS OF MATRIX ARE NORMALIZED
      DD = D(1,1)
      DO 23 M = 1,16
      DO 23 N = 1,16
23    D(M,N) = D(M,N)/DD
      CALL JACOBI (16,D,1,NR,Z)
      DO 999 N=1,16
999   ALAM(N) = SQRT(D(N,N)/PI**4**DD)
      PRINT3, (ALAM(N), N = 16)
3     FORMAT (2X, 16F8.3)
1000  CONTINUE
```

```
        IF(R.LT.3.5) GO TO 4
        IF(THETA.LT.90.) GO TO 31
        STOP
        END
$IBFTC  JACOBI NODECK
        SUBROUTINE JACOBI(N,Q,JVEC, M,V)
C       THIS SUB PROGRAM CALCULATES THE N EIGENVALUES,
        AND EIGENVECTORS OF A SYMMETRIC MATRIX BY A
        METHOD OF DIAGONALISATION.
        DIMENSION Q(16,16), V(16,16), X(16), IJ (16)
10      DO 14 I = 1,N
        DO 14 J = 1, N
        IF (I-J) 12, 11, 12
11      V(I,J) = 1,0
        GO TO 14
12      V(I,J) = 0.0
14      CONTINUE
15      M = 0
17      MI = N-1
        DO 30 I = 1, MI
        X(I) = 0.0
        MJ = I + I
        DO 30 J = MJ, N
        IF (X(I) - ABS(Q(I,J))) 20,20, 30
20      X(I) = ABS (Q(I,J))
        IH (I) = J
30      CONTINUE
40      DO 70 I = 1, MI
        IF (I - 1) 60,60, 45
45      IF (XMAX - X(I) 60,70,70
```

```
60        XMAX = X(I)
          IP = I
          JP = IH (I)
70        CONTINUE
          EPSI = 1.0E-8
          IF(XMAX - EPSI) 1000, 1000, 148
148       M = M + 1
          IF (Q(IP,IP) - Q(JP,JP) 150,151,151
150       TANG = -2,*Q(IP,JP)/(ABS(Q(IP,IP)-Q(JP,JP)
                         +SQRT(Q(IP,IP)-
     1    Q(JP,JP))**2 + 4.*Q(IP,JP)**2))
          GO TO 160
151       TANG = 2.*Q(IP,JP)/(ABS(Q(IP,IP)-Q(JP,JP))+SQRT
                          (Q(IP,IP)
     1    - Q(JP,JP)**2+4.*Q(IP,JP)**2))
160       COSN=1.0/SQRT(1.0+TANG**2)
          SINE =TANG*COSN
          QII = Q(IP,IP)
          Q(IP,IP) = COSN**2*(QII+TANG*(2.*Q(IP,JP)+TANG*Q
                                   (JP,JP)))
          Q(JP,JP) = COSN**2*(Q(JP,JP) - TANG*(2.Q(IP,JP)-
                               TANG*QII))
          Q(IP,JP) = 0.0
          IF(Q(IP,IP) - Q(JP,JP) 152,153,153
152       TEMP = Q(IP,IP)
          Q(IP,IP) = Q(JP,JP)
          Q(JP,JP) = TEMP
          IF(SINE) 154,155,155
154       TEMP = + COSN
          GO TO 170
155       TEMP = -COSN
```

```
170       COSN = ABS(SINE)
          SINE = TEMP
153       DO 350 I = 1,MI
          IF (I - IP) 210, 350, 200
200       IF (I- JP) 210, 350, 210
210       IF(IH(I) - IP) 230,240,230
230       IF(IIH(I) - JP) 350,240,350
240       K = IH(I)
250       TEMP = Q(I,K)
          Q(I,K) = 0.0
          MJ = I + 1
          X(I) = 0.0
          DO 320 J = MJ,N
          IF(X(I) - ABS(Q(I,J)))300,300,320
300       X(I) = ABS(Q(I,J))
          IH(I) = J
320       CONTINUE
          Q(I,K) = TEMP
350       CONTINUE
          X(IP) = 0.0
          X(JP) = 0.0
          DO 530 I = 1,N
          IF (I - IP) 370,530,420
370       TEMP = Q(I,IP)
          Q(I,IP) = COSN*TEMP + SINE*Q(I,JP)
          IF(X(I)-ABS(Q(I,IP)))380,390,390
380       X(I) = ABS(Q(I,IP))
          IH(I) = IP
```

```
390     Q(I,JP) = SINE*TEMP + COSN*Q(I,JP)
        IF(X(I)-ABS(Q(I,JP)))400,530,530
400     X(I) = ABS(Q(I,JP))
        IH(I) = JP
        GO TO 530
420     IF(I - JP)430,530,480
430     TEMP = Q(IP,I)
        Q(IP,I) = COSN*TEMP +SINE*Q(I,JP)
        IF(X(IP)-ABS(Q(IP,I)))440,450,450
440     X(IP)= ABS(Q(IP,I))
        IH(IP) = I
450     Q(I,JP) = -SINE*TEMP + COSN*Q(I,JP)
        IF(X(I)-ABS(Q(I,JP)))400,530,530
480     TEMP = Q(IP,I)
        Q(IP,I) = COSN*TEMP +SINE*Q(JP,I)
        IF(X(IP)-ABS(Q(IP,I)))490,500,500
490     X(IP) = ABS(Q(IP,I))
        IH(IP) = I
500     Q(JP,I) = -SINE*TEMP +COSN*Q(JP,I)
        IF(X(JP)-ABS(Q(JP,I)))510,530,530
510     X(JP) = ABS(Q(JP,I))
        IH(JP) = I
530     CONTINUE
        IF(JVEC) 540,40,540
540     DO 550 I = 1,N
        TEMP = V(I,IP)
        V(I,IP) = COSN*TEMP + SINE*V(I,JP)
550     V(I,JP) = -SINE*TEMP +COSN*V(I,JP)
        GO TO 40
1000    RETURN
        END
$ENTRY
```

APPENDIX - C

TABLES

INTEGRALS OF CHARACTERISTIC BEAM FUNCTIONS

m	n	CLAMPED-CLAMPED $\int_0^l \phi_m'\phi_n' dx$	$\int_0^l \phi_m'\phi_n dx$	$\int_0^l \phi_m'''\phi_n dx$	$\int_0^l \phi_m''\phi_n dx$	CLAMPED-SUPPORTED $\int_0^l \phi_m'\phi_n' dx$	$\int_0^l \phi_m'\phi_n dx$	$\int_0^l \phi_m'''\phi_n dx$	$\int_0^l \phi_m''\phi_n dx$
1	1	12.303	0.000	0.000	-12.303	11.513	0.000	16.303	-11.513
1	2	0.000	3.344	122.035	0.000	-4.262	2.997	62.328	4.262
1	3	-9.730	0.000	0.000	9.730	-3.798	-0.514	10.635	3.798
1	4	0.000	0.907	58.500	0.000	-3.268	1.004	26.000	3.268
2	1	0.000	-3.344	-122.035	0.000	-4.262	-2.995	-119.341	4.262
2	2	46.050	0.000	.000	-46.050	42.397	0.000	49.048	-42.397
2	3	0.000	5.516	476.767	0.000	-7.820	5.139	512.557	7.820
2	4	-17.236	0.000	0.000	17.236	-7.664	-0.254	41.201	7.664
3	1	-9.730	0.000	0.000	9.730	-3.798	0.517	71.619	3.798
3	2	0.000	-5.516	-476.767	0.000	-7.820	-5.147	-456.721	7.820
3	3	98.900	0.000	0.000	-98.900	94.012	0.000	104.248	-94.012
3	4	0.000	7.032	1130.357	0.000	-11.152	7.258	858.752	11.152
4	1	0.000	-0.907	-58.500	0.000	-3.268	-1.009	-135.024	3.268
4	2	-17.200	0.000	0.000	17.200	-7.664	0.254	147.582	7.664
4	3	0.000	-7.032	-1130.357	0.000	-11.152	-7.258	-1141.599	11.152
4	4	171.500	0.000	0.000	-171.500	164.888	0.000	176.340	-164.888

INTEGRALS OF CHARACTERISTIC BEAM FUNCTIONS

		CLAMPED-FREE				FREE-SUPPORTED			
m	n	$\int_0^l \phi_m' \phi_n' dx$	$\int_0^l \phi_m' \phi_n dx$	$\int_0^l \phi_m'' \phi_n' dx$	$\int_0^l \phi_m'' \phi_n dx$	$\int_0^l \phi_m' \phi_n' dx$	$\int_0^l \phi_m' \phi_n dx$	$\int_0^l \phi_m'' \phi_n' dx$	$\int_0^l \phi_m'' \phi_n dx$
1	1	4.648	2.000	3.700	0.858	3.000	-1.500	0.000	0.000
1	2	-7.580	0.759	4.988	1.874	3.464	0.841	0.000	0.000
1	3	3.982	0.216	2.336	1.565	3.464	-0.346	0.000	0.000
1	4	-3.594	0.120	1.815	1.088	3.464	0.240	0.000	0.000
2	1	-7.300	-4.759	-81.320	-11.745	3.464	-4.106	-22.288	10.149
2	2	32.417	2.000	45.709	-13.298	27.251	-2.000	-16.502	-11.512
2	3	-22.352	2.224	85.585	3.229	14.397	2.004	36.828	1.524
2	4	13.502	0.818	32.420	5.541	15.157	-0.792	-10.821	0.582
3	1	3.942	3.704	40.065	27.455	3.464	-6.118	-7.151	21.023
3	2	-22.852	-6.219	-235.472	-9.010	14.397	-6.008	-201.983	13.077
3	3	77.300	2.000	123.804	-45.904	71.171	-1.995	-49.848	-42.898
3	4	-35.848	4.108	330.713	4.280	24.527	3.855	218.894	3.748
4	1	-3.604	-4.422	-62.309	-37.411	3.464	-3.703	-60.370	31.905
4	2	13.582	3.365	177.986	50.415	15.157	-3.203	-71.090	25.684
4	3	-35.048	-8.172	-705.121	-8.545	24.527	-7.855	-652.104	18.318
4	4	142.902	2.000	242.035	-98.904	154.665	-2.000	-104.249	-94.054

INTEGRALS OF CHARACTERISTIC BEAM FUNCTIONS

m	n	FREE - FREE $\int_0^l \phi_m' \phi_n' dx$	$\int_0^l \phi_m' \phi_n dx$	$\int_0^l \phi_m'' \phi_n' dx$	$\int_0^l \phi_m'' \phi_n dx$	SUPPORTED - SUPPORTED $\int_0^l \phi_m' \phi_n' dx$	$\int_0^l \phi_m' \phi_n dx$	$\int_0^l \phi_m'' \phi_n' dx$	$\int_0^l \phi_m'' \phi_n dx$
1	1	0.000	0.000	0.000	0.000	9.870	0.000	0.000	-9.870
1	2	0.000	0.000	0.000	0.000	0.000	2.667	26.319	0.000
1	3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	00.000
1	4	0.000	0.000	0.000	0.000	0.000	1.067	10.528	0.000
2	1	0.000	-3.464	0.000	0.000	0.000	-2.667	-105.276	0.000
2	2	12.000	0.000	0.000	0.000	39.479	0.000	0.000	-39.479
2	3	0.000	0.000	0.000	0.000	0.000	4.800	189.495	0.000
2	4	15.856	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	1	0.000	0.000	0.000	10.589	0.000	0.000	0.000	00.000
3	2	0.000	-6.928	-34.835	0.000	0.000	-4.800	-426.372	0.000
3	3	49.481	0.000	0.000	-12.301	88.827	0.000	0.000	-88.827
3	4	0.000	1.214	44.282	0.000	0.000	6.858	609.086	0.000
4	1	0.000	-4.000	0.000	0.000	0.000	-1.067	-168.446	0.000
4	2	13.856	0.000	0.000	40.594	0.000	0.000	0.000	0.000
4	3	0.000	-9.212	-333.476	0.000	0.000	-6.858	-1082.836	0.000
4	4	108.925	0.000	0.000	-46.050	157.914	0.000	0.000	-157.914

TABLE 3

Fundamental frequencies $\left[\lambda = (\rho h w^2 a^2 b^2 / \pi^4 D_{126}')^{1/2}\right]$ of specially orthotropic SSSS plates (Maple plywood) $D_{11}' / D_{126}' = 1.545$, $D_{22}' / D_{126}' = 4.815$)

Side ratio a/b	ANGLE OF ORTHOTROPICITY DEGREES			
	0		90	
	From equation 2.20	by Raleigh's Method Hearmon (Ref. 5)	From equation 2.20	by Raleigh's Method Hearmon (Ref. 5)
0.4	3.523	3.523	5.686	5.686
0.6	2.832	2.832	3.991	3.991
0.8	2.757	2.757	3.242	3.242
1.0	2.891	2.891	2.891	2.891
1.2	3.165	3.165	2.750	2.750
1.4	3.496	3.496	2.735	2.735
1.6	3.835	3.835	2.798	2.798
1.8	4.251	4.251	2.915	2.915
2.0	4.652	4.652	3.062	3.062
2.2	5.031	5.031	3.235	3.235
2.4	5.476	5.476	3.424	3.424
2.6	5.896	5.896	3.625	3.625
2.8	6.~~294~~ 319	6.319	3.835	3.835
3.0	6.745	6.745	4.052	4.052
3.2	7.172	7.172	4.274	4.274
3.4	7.601	7.601	4.500	4.500
3.6	8.031	8.031	4.750	4.750
3.8	8.462	8.462	4.961	4.961
4.0	8.894	8.894	5.195	5.195

TABLE 4

Natural frequencies $\left[\lambda = (\rho h w^2 a^2 b^2 / \pi^4 D'_{126})^{1/2}\right]$ of generally orthotropic SSSS plate (Maple) plywood plate, $D'_{11} / D'_{126} = 1.545$; $D'_{22}/D'_{126} = 4.813$)

Side Ratio a/b	m	n	0	15	30	45	60	75	90
			ANGLE OF ORTHOTROPICITY DEGREES						
0.4	1	1	3.525	3.689	4.095	4.557	5.047	5.496	5.686
	2	1	12.770	12.810	13.462	15.568	18.440	21.076	22.126
	3	1	28.283	28.085	29.087	33.600	40.735	47.085	49.546
	4	1	50.016	49.422	50.845	59.191	72.278	85.568	89.958
	1	2	5.474	5.879	6.587	6.865	6.755	6.556	6.485
	1	3	9.486	9.762	10.216	10.150	9.481	8.617	8.251
1.0	1	1	2.891	3.022	3.292	3.427	3.292	3.022	2.891
	2	1	6.124	6.432	7.035	8.795	8.899	9.154	9.506
	3	1	12.153	12.707	12.239	16.628	18.490	19.506	20.234
	4	1	20.780	21.086	22.254	26.145	30.798	34.951	35.576
	1	2	9.506	9.154	8.899	7.333	7.035	6.432	6.124
	1	3	20.234	19.506	18.490	15.256	12.239	12.707	12.157
1.6	1	1	3.885	3.840	3.801	3.697	3.408	3.006	2.798
	2	1	5.474	5.755	6.361	6.815	6.886	6.629	6.485
	3	1	8.898	9.349	10.339	11.255	11.743	13.338	13.200
	4	1	14.093	14.641	15.983	20.879	21.743	22.542	22.743
	1	2	14.344	13.780	12.386	10.861	9.596	8.737	8.549
	1	3	31.885	30.408	26.859	22.468	19.564	18.524	18.435

Continued

Side Ratio a/b	m	n	ANGLE OF ORTHOTROPICITY DEGREES 0	15	30	45	60	75	90
2.0	1	1	4.652	4.548	4.315	4.010	3.673	3.258	3.082
	2	1	5.781	5.875	6.416	6.740	6.609	6.087	5.781
	3	1	8.278	8.721	9.705	10.338	11.249	10.985	11.029
	4	1	12.248	12.916	14.363	16.273	16.851	19.136	18.307
	1	2	17.783	17.005	15.100	12.822	10.881	10.552	10.390
	1	3	39.722	37.783	32.902	27.788	23.674	22.715	22.785
3.0	1	1	6.745	6.480	5.834	5.134	4.470	4.134	4.032
	2	1	7.355	7.516	7.365	7.180	6.752	5.955	5.518
	3	1	8.672	8.951	9.536	10.077	9.916	9.141	8.678
	4	1	10.918	11.635	13.098	14.173	14.822	13.855	13.520
	1	2	26.481	25.185	21.894	19.207	15.825	15.155	15.190
	1	3	59.367	56.367	48.645	39.900	34.456	33.455	33.814
4.0	1	1	8.894	8.498	7.489	6.588	5.619	5.267	5.135
	2	1	9.306	9.078	8.559	8.052	7.372	6.351	6.124
	3	1	10.140	10.176	10.828	10.427	9.944	8.829	8.159
	4	1	11.163	12.122	13.229	13.873	13.470	12.266	11.565
	1	2	35.217	33.459	28.894	25.762	20.535	19.901	20.058
	1	3	78.038	75.016	64.584	52.673	45.409	44.313	45.822

TABLE 5

Buckling loads and Natural Frequencies of Specially orthotropic SSSS plates (Maple) plywood plate $D'_{11} / D'_{126} = 1.545$, $D'_{22} / D'_{126} = 4.815$)

R_1 $= \frac{N_1^2 a^2}{\pi^2 D'_{11}}$	m	n	R_2 ($= N_2 a^2/\pi^2 D'_{11}$) Side Ratio 0.5	Side Ratio 1.0	Side Ratio 2.0
-8	1	1	9.092	9.615	15.395
	1	2	2.404	3.848	24.381
	2	1	48.507	49.475	57.856
-4	1	1	5.092	5.615	11.395
	1	2	1.404	2.848	23.381
	2	1	32.507	33.475	41.856
0	1	1	1.092	1.615	7.395
	1	2	0.404	1.848	22.381
	2	1	16.507	17.475	25.856
4	1	1	-2.908	-2.385	3.395
	1	2	-0.596	0.848	21.381
	2	1	0. 507	1.475	9.856
8	1	1	-6.908	-6.385	-0.607
	1	2	-1.596	-0.152	20.381
	2	1	-15.693	-14.525	-6.164

Natural frequency = $\lambda = (R_2\, \pi\, D'_{11} / \pi^2 D'_{126})^{1/2}$

TABLE 6a

Frequencies $\left[\lambda = (\rho h\,\omega^2 a^4 / \pi^4 D'_{22})^{1/2}\right]$ of generally orthotropic CCCC plates (Maple plywood, $D'_{11} / D'_{22} = 3.117$, $D'_{66} / D'_{22} = .262$, $D'_{12} / D'_{22} = 0.12$)

Side ratio		ANGLE OF ORTHOTROPICITY DEGREES						
		0	15	30	45	60	75	90
0.5	1	4.101	3.942	3.522	3.028	2.687	2.579	2.576
	2	4.509	4.407	4.329	4.086	3.773	3.735	3.823
	3	5.427	5.529	5.860	5.520	5.260	5.818	6.186
	4	6.974	7.182	7.598	7.560	6.684	6.457	6.471
	5	11.151	10.617	9.298	8.005	7.801	7.661	7.323
	6	11.452	11.110	10.255	9.180	8.271	8.148	9.154
1.0	1	4.813	4.765	4.609	4.545	4.609	4.763	4.813
	2	7.907	7.985	8.255	8.490	8.255	7.985	7.907
	3	11.589	11.208	10.379	9.877	10.379	11.208	11.589
	4	13.471	13.398	13.258	13.367	13.256	13.398	13.491
	5	13.738	14.077	14.887	15.357	14.887	14.077	13.738
	6	18.228	18.547	18.957	17.282	18.957	18.547	18.228
1.5	1	6.821	6.819	7.008	7.597	8.470	9.251	9.527
	2	12.828	12.318	11.717	11.845	11.905	11.792	11.705
	3	13.170	15.389	16.196	17.818	17.475	16.742	16.382
	4	19.582	19.697	17.980	18.324	21.[illegible] 456	23.975	23.528
	5	23.018	22.828	23.070	23.850	23.248	24.205	25.202
	6	26.306	25.795	26.019	26.097	25.505	26.587	26.798
2.0	1	10.309	10.318	10.748	12.112	14.087	15.769	16.417
	2	15.288	14.910	15.061	16.255	17.314	17.860	18.058
	3	24.785	22.471	21.041	22.029	22.839	22.115	21.707
	4	25.885	25.850	26.755	30.239	30.391	28.729	27.885
	5	29.290	30.644	31.208	32.019	37.194	42.469	44.550
	6	33.538	32.591	33.084	36.639	40.941	44.484	45.808

Side Ratio a/b	Mode No.	ANGLE OF ORTHOTROPICITY DEGREES 0	15	30	45	60	75	90
2.5	1	15.137	15.111	15.750	18.002	21.375	24.256	25.369
	2	19.214	18.051	18.825	22.136	24.478	26.108	26.678
	3	27.736	25.656	25.148	27.818	29.615	29.789	29.824
	4	39.852	36.995	35.581	37.157	37.210	35.676	34.805
	5	40.687	39.826	41.338	47.519	57.366	65.989	69.341
	6	42.667	43.679	46.367	55.056	61.145	67.896	70.491
3.0	1	21.214	21.106	21.896	25.233	30.302	34.661	36.348
	2	24.600	24.570	25.912	29.388	35.553	36.353	37.474
	3	32.084	30.295	30.311	33.096	38.356	39.654	39.955
	4	44.111	41.903	41.493	44.594	44.765	44.920	44.286
	5	56.962	56.471	58.311	67.471	82.084	94.744	100.759
	6	59.442	60.172	65.926	73.106	86.326	96.600	102.796
3.5	1	28.482	28.286	29.823	35.795	40.836	46.974	49.341
	2	31.572	31.534	33.234	37.979	43.849	49.589	50.352
	3	37.910	36.495	37.934	45.742	48.805	51.604	52.489
	4	48.909	47.665	49.869	55.355	56.048	56.396	56.220
	5	77.220	76.368	78.565	91.090	111.169	128.751	135.545
	6	79.490	79.859	84.164	96.776	114.990	130.686	136.576

TABLE 6 b

Frequencies $\left[\lambda=(\rho h\, \omega^2 a^4/\pi^4 D'_{22})^{1/2}\right]$ of generally orthotropic SSCC plate
(Maple plywood $D'_{11}/D'_{22} = 3.117$, $D'_{66}/D'_{22} = 262$; $D'_{12}/D'_{22} = 0.12$)

Side Ratio a/b	Mode No.	ANGLE OF ORTHOTROPICITY (DEGREES) 0	15	30	45	60	75	90
0.5	1	1.959	1.948	1.900	1.800	1.669	1.573	1.550
	2	2.657	2.791	3.030	3.111	2.939	3.057	3.181
	3	3.939	4.203	4.619	4.703	4.517	4.378	4.310
	4	5.084	6.141	6.156	5.280	4.634	4.883	5.411
	5	7.136	6.192	6.754	6.078	6.412	6.299	5.789
	6	7.654	7.590	7.587	7.018	6.808	7.012	7.635
1.0	1	3.141	3.229	3.462	3.753	4.023	4.252	4.318
	2	6.945	7.020	7.018	6.910	6.651	6.545	6.200
	3	7.855	7.824	8.124	8.910	9.853	10.744	10.522
	4	10.629	10.909	11.545	11.451	11.175	10.956	10.552
	5	12.881	13.000	13.941	13.651	13.037	12.784	12.736
	6	13.877	13.857	14.972	16.200	16.814	16.356	16.029
1.5	1	5.725	5.266	6.250	7.079	8.103	8.941	9.263
	2	9.504	9.481	9.745	10.303	10.640	10.851	10.567
	3	14.642	14.758	13.125	13.558	13.141	14.592	13.951
	4	17.395	15.958	15.651	17.957	21.148	20.516	19.863
	5	17.686	19.094	20.769	22.560	22.053	24.050	25.127
	6	25.214	25.229	25.142	23.093	24.853	25.699	26.215
2.0	1	9.581	9.645	10.214	11.750	13.817	15.572	16.248
	2	12.863	12.693	13.526	15.026	16.324	17.053	17.272
	3	19.907	18.763	18.646	20.032	20.682	20.257	19.885
	4	23.535	23.419	26.317	26.122	27.410	25.699	24.801
	5	27.771	28.605	29.154	30.858	36.988	42.342	44.454
	6	31.339	29.857	31.542	35.518	40.196	45.957	45.406

Side Ratio a/b	Mode No.	ANGLE OF ORTHOTROPICITY						
		0	15	30	45	60	75	90
2.5	1	14.827	14.821	15.325	17.695	21.153	24.105	25.245
	2	17.055	17.305	18.595	21.065	23.660	25.478	26.127
	3	23.315	22.557	25.406	26.159	27.040	28.510	28.247
	4	34.058	32.692	32.905	34.521	34.544	33.135	32.295
	5	39.576	39.334	40.773	47.186	57.170	65.881	69.265
	6	41.547	42.404	45.606	52.054	60.469	67.487	70.185
3.0	1	20.850	20.728	21.569	24.268	30.111	34.538	36.252
	2	22.894	23.195	24.867	28.452	32.626	35.048	37.051
	3	28.266	27.550	29.548	33.618	36.672	38.442	38.882
	4	38.014	37.813	39.148	42.144	45.362	42.786	42.266
	5	56.748	56.185	57.926	67.169	81.851	94.647	100.507
	6	58.568	59.152	62.818	72.169	85.200	96.245	102.188
3.5	1	28.179	27.859	29.949	35.557	40.694	46.858	49.262
	2	30.061	30.295	32.294	37.118	43.217	49.140	50.011
	3	34.657	34.516	36.977	42.421	47.450	50.575	51.622
	4	43.404	43.715	46.551	51.050	53.349	54.566	54.568
	5	77.050	76.131	78.235	90.309	111.012	128.640	135.467
	6	78.775	79.025	85.153	95.394	114.595	130.251	136.559

TABLE 6c

Frequencies $\lambda = \left[(\rho h \omega^2 a^4 / \pi^4 D'_{22})^{1/2} \right]$ of generally orthotropic SSSS plate (Maple plywood, D'_{11}/D'_{22} = 3.117, D'_{66}/D'_{22} = .282, D'_{12}/D'_{22} = 0.12)

Side Ratio a/b	Mode No.	ANGLE OF ORTHOTROPICITY DEGREES 0	15	30	45	60	75	90
0.5	1	1.873	1.842	1.745	1.581	1.379	1.250	1.232
	2	2.327	2.454	2.600	2.580	2.347	2.250	2.327
	3	3.331	3.354	3.911	3.959	3.780	4.009	4.182
	4	4.950	5.225	5.324	5.161	4.485	4.217	4.440
	5	7.161	6.865	6.108	6.330	5.895	5.259	4.950
	6	7.490	7.390	[illegible]	[illegible]	[illegible]	6.381	6.592
1.0	1	2.327	2.358	2.472	2.553	2.477	2.357	2.327
	2	4.950	5.018	5.291	5.659	5.659	5.174	4.950
	3	7.490	7.381	7.073	6.853	6.841	7.252	7.490
	4	9.309	9.087	9.210	9.651	9.908	9.609	9.309
	5	9.707	10.309	11.126	12.047	11.128	10.070	9.707
	6	13.326	13.502	13.969	13.190	14.057	14.116	13.326
1.5	1	3.331	3.389	3.723	4.145	4.372	4.435	4.440
	2	6.184	7.349	7.281	7.654	7.748	7.020	6.592
	3	9.767	10.167	10.957	12.365	12.909	11.781	11.032
	4	13.326	12.401	11.708	12.775	14.136	15.676	16.231
	5	16.653	17.057	17.694	18.004	17.967	17.671	17.759
	6	20.945	19.031	18.957	20.342	20.245	18.804	17.845
2.0	1	4.950	5.003	5.529	6.386	6.965	7.368	7.490
	2	9.309	8.757	8.902	10.299	10.559	9.797	9.309
	3	16.729	14.487	13.201	15.512	15.904	14.350	13.326
	4	17.759	16.775	17.505	20.426	23.559	21.041	19.720
	5	19.720	21.041	22.905	24.576	24.427	27.427	28.644
	6	23.369	23.003	24.453	26.524	28.345	29.571	29.051

Side ratio	Mode No.	ANGLE OF ORTHOTROPICITY DEGREES						
		0	15	30	45	60	75	90
2.5	1	7.091	7.189	7.844	9.088	10.287	11.141	11.444
	2	11.017	10.547	11.219	13.374	14.024	15.473	13.049
	3	13.097	15.822	15.403	18.859	19.610	17.808	16.607
	4	25.701	24.691	24.710	28.151	27.727	24.353	22.527
	5	28.384	27.325	28.336	31.601	37.294	42.307	44.555
	6	31.103	30.187	33.488	37.345	41.487	44.836	45.778
3.0	1	9.787	9.853	10.657	12.365	14.289	15.750	16.291
	2	13.396	13.070	14.249	17.022	18.184	18.071	17.759
	3	20.945	17.952	18.353	22.848	24.002	22.194	20.945
	4.	32.655	28.909	29.418	33.094	32.429	28.517	26.370
	5.	36.063	36.813	38.509	44.191	53.120	60.951	63.982
	6	39.148	40.708	44.560	50.411	57.450	63.051	65.182
3.5	1	13.002	13.048	13.969	16.226	18.928	21.195	22.024
	2	16.232	16.025	17.881	21.211	25.020	25.425	25.406
	3	23.380	20.710	22.074	27.429	29.038	27.471	25.507
	4	34.682	32.262	33.908	38.320	37.753	33.606	31.277
	5	49.673	49.400	51.355	59.365	71.803	82.397	86.923
	6	52.009	53.355	57.680	65.757	76.197	84.814	88.097
4.0	1	16.729	16.745	17.780	20.853	24.404	27.475	28.844
	2	19.720	19.850	22.028	25.928	28.536	29.678	29.861
	3	30.369	24.220	26.449	32.363	34.788	33.627	32.855
	4	37.236	35.803	38.777	43.985	45.730	39.588	37.286
	5	64.639	64.155	66.304	78.708	95.586	107.908	113.417
	6	66.916	67.901	72.839	83.559	97.805	109.825	114.574

TABLE 64

Frequencies $[\lambda = (\rho h \omega^2 a^4/\pi^4 D'_{22})^{1/2}$ of generally orthotropic SSSF plate (Maple plywood).

Side ratio a/b	Mode No.	ANGLE OF ORTHOTROPICITY 0	15	30	45	60	75	90
0.5	1	1.789	1.710	1.507	1.271	1.105	1.045	1.058
	2	1.965	2.028	2.087	2.055	1.819	1.555	1.402
	3	2.622	2.885	3.159	4.695	3.059	2.855	2.857
	4.	3.788	4.107	4.692	5.269	4.101	3.996	4.041
	5	7.005	6.741	5.848	5.155	4.964	4.585	4.584
	6	7.278	7.054	6.490	5.784	5.114	4.884	5.192
1.0	1	1.862	1.790	1.654	1.518	1.370	1.214	1.145
	2	2.685	2.859	3.211	3.447	3.591	2.945	2.866
	3	6.116	6.215	5.827	5.170	4.476	4.186	4.152
	4	7.156	6.914	6.851	7.040	6.918	6.152	5.607
	5	7.911	8.056	8.400	8.805	9.071	9.105	9.159
	6	10.490	10.735	11.129	10.897	9.560	9.330	9.487
1.5	1	1.954	1.881	1.755	1.808	1.722	1.455	1.307
	2	3.874	3.827	3.953	4.369	4.682	4.475	4.335
	3	7.282	2.017	6.699	6.486	5.835	5.121	4.795
	4	9.001	8.622	8.104	8.888	9.309	8.717	7.701
	5	12.543	12.741	13.157	12.605	10.894	9.420	9.345
	6	16.018	14.255	13.767	14.894	15.513	14.050	12.616
2.0	1	2.021	1.852	1.862	2.078	2.118	1.740	1.506
	2	5.558	4.555	3.668	4.781	5.299	4.872	4.562
	3	7.408	7.581	6.965	8.562	7.508	8.056	7.767
	4	10.753	8.605	7.006	8.801	10.131	9.842	9.600
	5	16.250	15.688	14.516	15.807	14.391	11.976	10.664
	6	19.469	16.610	16.701	16.379	16.997	16.687	15.539

Side ratio a/b	Mode No.	ANGLE OF ORTHOTROPICITY						
		0	15	30	45	60	75	90
2.5	1	2.255	1.658	———	2.286	2.583	2.051	1.728
	2	7.727	4.444	1.272	4.971	5.974	5.588	4.885
	3	7.594	7.851	5.809	8.474	10.882	10.386	9.324
	4	12.838	8.711	8.588	10.971	11.789	11.735	11.636
	5	16.459	16.211	14.885	16.595	17.989	15.935	14.495
	6	21.563	18.212	20.159	20.351	18.545	17.274	16.941
3.0	1	2.442	———	———	2.359	2.935	2.376	1.967
	2	7.817	2.347	———	4.905	6.714	5.854	5.229
	3	10.583	5.099	1.470	8.015	11.725	10.974	10.309
	4	15.498	9.987	9.101	13.245	15.656	16.222	16.342
	5	16.869	16.582	16.454	17.681	19.017	17.955	17.340
	6	24.149	20.919	24.094	24.841	25.208	20.621	19.185
3.5	1	2.846	———	———	1.760	3.405	2.712	2.215
	2	8.075	———	———	4.649	7.498	6.418	5.611
	3	13.552	1.915	———	7.500	12.881	11.854	10.748
	4	18.958	10.975	9.958	14.981	19.421	18.875	17.904
	5	18.615	17.592	19.075	20.500	20.980	21.519	21.904
	6	27.229	24.194	28.372	29.697	28.404	26.059	24.757

TABLE 6e

Frequencies $\left[\lambda = (\rho h \omega^2 a^4/\pi^4 D_{22}')^{1/2}\right]$ of generally orthotropic CFFF plate

(Maple plywood, $D_{11}'/D_{22}' = 3.117$, $D_{66}'/D_{22}' = .232$, $D_{12}'/D_{22}' = 0.12$)

Side ratio a/b	Mode No.	ANGLE OF ORTHOTROPICITY						
		0	15	30	45	60	75	90
0.5	1	0.629	0.590	0.505	0.410	0.355	0.349	0.356
	2	0.731	0.756	0.775	0.726	0.589	0.500	0.515
	3	1.155	1.240	1.365	1.[illegible]	1.174	1.183	1.280
	4	2.001	2.124	2.320	2.354	2.193	2.195	2.253
	5	3.943	3.715	3.175	2.597	2.315	2.394	2.436
	6	4.033	3.959	3.618	3.257	2.865	2.939	2.957
1.0	1	0.629	0.563	0.430	0.366	0.350	0.348	0.356
	2	0.936	1.053	1.177	1.220	1.098	0.887	0.765
	3	2.740	2.840	2.881	2.390	2.143	2.176	2.250
	4	3.943	3.603	3.194	3.250	3.240	3.114	2.990
	5	4.408	4.330	4.357	4.394	4.383	4.265	4.251
	6	6.147	6.486	6.776	6.260	6.215	6.165	6.078
1.5	1	0.629	0.490	0.351	0.331	0.344	0.347	0.356
	2	1.259	1.279	1.389	1.557	1.502	1.259	1.031
	3	3.959	3.406	2.672	[illegible]	2.142	2.178	2.250
	4	4.944	4.690	4.411	4.750	5.000	4.217	3.705
	5	5.823	5.678	5.675	6.125	6.000	6.115	6.243
	6	8.856	8.277	7.821	7.940	8.111	8.349	7.811
2.0	1	0.629	0.275	———	0.205	0.345	0.348	0.356
	2	1.5[illegible]	1.031	———	1.227	1.331	1.619	1.556
	3	3.940	2.936	1.765	2.514	2.226	2.179	2.250
	4	5.597	4.625	2.855	4.007	5.670	5.228	4.474
	5	9.450	8.477	6.904	7.885	8.805	6.159	6.245
	6	11.011	9.479	9.[illegible]	9.735	11.017	8.959	8.945

Side Ratio a/b	Mode No.	ANGLE OF ORTHOTROPICITY 0	15	30	45	60	75	90
2.5	1	0.629	...	...	...	0.323	0.344	0.356
	2	1.791	...	...	...	1.968	1.955	1.614
	3	3.940	1.882	...	1.019	2.651	2.204	2.250
	4	6.317	3.287	2.197	2.856	5.565	5.330	5.270
	5	11.029	7.716	5.821	7.910	8.255	6.581	6.245
	6	13.517	11.374	11.200	10.565	11.266	11.245	9.999
3.0	1	0.629	...	...	...	0.308	0.343	0.356
	2	2.086	...	...	...	1.891	2.101	1.895
	3	3.940	...	...	...	3.138	2.414	2.250
	4	7.074	2.280	1.492	3.080	5.344	5.854	6.062
	5	11.051	6.668	5.237	6.850	9.660	7.567	6.245
	6	14.489	12.176	12.662	12.068	11.179	11.658	11.225
3.5	1	0.629	...	...	...	0.286	0.340	0.356
	2	2.362	...	...	...	1.762	2.110	2.175
	3	3.940	...	...	...	3.630	2.762	2.250
	4	7.852	1.751	0.811	2.617	5.051	5.922	6.244
	5	11.051	5.926	5.518	5.921	10.577	8.642	6.906
	6	15.552	13.032	14.213	13.755	11.565	11.699	12.257

TABLE 7

Frequencies $\left[\lambda = (\rho h \omega^2 a^4 / \pi^4 D'_{22})^{1/2}\right]$ of specially orthotropic plates (Maple plywood $D'_{11}/D'_{22} = 3.117$; $D'_{12}/D'_{22} = .12$ $D'_{66}/D'_{22} = .262$)

Angle of ortho-tropicity θ degrees	Side Ratio a/b	CCCC From Eqn. 2.31	CCCC By Raleigh's Method	SSCC From Eqn. 2.31	SSCC By Raleigh's Method	SSSS From Eqn. 2.31	SSSS By Raleigh's Method
0	0.5	4.104	4.114	1.959	1.959	1.873	1.871
	1.0	4.813	4.817	3.141	3.140	2.327	2.326
	1.5	6.821	6.827	5.723	5.722	3.331	3.328
	2.0	10.306	10.321	9.581	9.519	4.930	4.926
	2.5	15.137	15.170	14.627	14.625	7.091	7.087
	3.0	21.214	21.269	20.830	20.828	9.787	9.783
	3.5	28.482	28.564	28.179	28.176	13.002	12.998
90	0.5	2.576	2.580	1.550	1.550	1.232	1.232
	1.0	4.813	4.817	4.318	4.316	2.327	2.326
	1.5	9.527	9.546	9.263	9.257	4.440	4.436
	2.0	16.417	16.457	16.248	16.239	7.490	7.485
	2.5	25.369	25.437	25.246	25.233	11.444	11.437
	3.0	36.348	36.451	36.262	36.236	16.291	16.281
	3.5	49.341	49.484	49.262	49.236	22.024	22.012

TABLE 8

Frequencies $\left[\lambda = (\rho h\omega^2 a^4/\pi^4 D'_{22})^{1/2}\right]$ of specially orthotropic cantilever (CFFF) plates (Maple plywood $D'_{11}/D'_{22} = 3.117$; $D'_{12}/D'_{22} = .12$; $D'_{66}/D'_{22} = .26$)

Side ratio a/b	Mode No.	from equation 2.31	from Reference 9
0.5	1	0.629	0.629
	2	0.731	0.730
	3	1.133	1.127
	4	2.001	2.000
	5	3.943	---
1.0	1	0.629	0.629
	2	0.966	0.966
	3	2.740	2.740
	4	3.945	3.942
	5	4.408	---
2.0	1	0.629	0.629
	2	1.239	1.195
	3	3.939	3.938
	4	4.944	---
	5	5.523	5.601

TABLE 9

Buckling loads ($k_b = N_1 b^2 / h^3 E_L$) of generally orthotropic SSSS plates (Mahogony plywood $D'_{11}/D'_{22} = 3.04$, $D'_{126}/D'_{22} = 0.438$, $E_L = 1.36 \times 10^6$ p.s.i.)

Side Ratio	m	n	ANGLE OF ORTHOTROPICITY, θ 0	15	30	45	60	75	90
0.4	1	1	4.408	4.090	3.431	2.806	2.269	1.833	1.670
	2	1	16.924	16.414	11.406	8.004	5.991	5.619	5.702
	3	1	37.824	36.886	31.287	16.352	12.321	12.043	12.541
1.0	1	1	1.080	1.209	1.494	1.645	1.494	1.209	1.080
	2	1	2.922	2.828	2.690	2.650	1.688	1.383	1.236
	3	1	6.237	5.764	4.822	3.463	2.284	2.064	2.240
1.6	1	1	1.015	1.200	1.598	1.892	1.966	1.975	1.990
	2	1	1.377	1.468	1.628	1.618	1.384	1.090	1.963
	3	1	2.606	2.669	2.693	1.931	1.621	1.294	1.153
2.0	1	1	1.236	1.420	1.843	2.234	2.519	2.789	2.922
	2	1	1.080	1.233	1.516	1.587	1.406	1.174	1.080
	3	1	1.796	1.789	1.809	1.71	1.430	1.115	0.983
2.4	1	1	1.571	1.740	2.174	2.679	3.229	3.812	4.083
	2	1	0.972	1.153	1.498	1.631	1.512	1.362	1.307
	3	1	1.377	1.470	1.653	1.612	1.363	1.079	0.963
3.2	1	1	2.503	2.616	3.053	3.855	5.093	6.453	7.063
	2	1	1.015	1.211	1.619	1.863	1.898	1.953	1.990
	3	1	1.029	1.197	1.604	1.585	1.414	1.218	1.146

TABLE 10

Normal ($K_b = N_1 b^2/h^3 E_L$) and Shear ($K_s = N_{12} b^2/h^3 E_L$) Buckling loads of square specially orthotropic SSSS plates (Mahogony plywood $D'_{11}/D'_{22} = 3.04$, $D'_{126}/D'_{22} = 0.438$, $E_L = 1.35 \times 10^6$ p.s.i

Angle of orthotropicity degrees	NORMAL BUCKLING CO-EFF. K_b		SHEAR BUCKLING COEFF. K_s	
	From Eqn. 2.20	From Reference 16	From equation 2.20	From reference 16
0	1.080	1.08	2.566	2.56
			-2.566	-2.56
15	1.209	1.21	2.241	2.24
			-3.607	-3.61
30	1.494	1.50	2.399	2.41
			-4.900	-4.89
45	1.645	1.65	2.526	2.53
			-5.511	-5.50
60	1.494	1.50	2.399	2.41
			-4.900	-4.89
75	1.209	1.21	2.241	2.24
			-3.607	-3.61
90	1.080	1.08	2.566	2.56
			-2.566	-2.56

TABLE 11

Shear Buckling loads $K_s = \frac{N_{12}b^2}{h^3 E_L}$ of generally orthotropic SSSS plates (Mahogony plywood $D'_{11}/D'_{22} = 3.04$, $D'_{126}/D'_{22} = 0.438$, $E_L = 1.36 \times 10^6$ p.s.i.)

Angle of orthotropicity degrees	SHEAR BUCKLING COEFFICIENTS K_s		
	a/b = 0.5	a/b = 1.0	a/b = 1.5
0	-9.731	-2.566	-1.579
	9.731	2.566	1.579
15	-13.243	-3.607	-2.127
	7.778	2.241	1.504
30	-15.392	-4.900	-3.120
	6.951	2.399	1.794
45	-13.805	-5.511	-3.969
	6.656	2.525	1.848
60	-10.200	-4.900	-4.046
	6.571	2.399	1.814
75	-7.000	-3.607	-3.394
	6.038	2.241	1.930
90	-5.735	-2.566	-2.464
	5.735	2.566	2.464

TABLE 12

Bucling loads ($K_b = N_1 b^2/\pi^2 D'_{22}$) of generally orthotropic plates with various boundary conditions (Maple plywood $D'_{11}/D'_{22} = 3.117$, $D'_{12}/D'_{22} = 0.12$, $D'_{66}/D'_{22} = .26$)

Boundary conditions	Side ratio a/b	ANGLE OF ORTHOTROPICITY 0	15	30	45	60	75	90
	0.5	52.571	48.403	38.429	28.247	22.181	20.542	20.621
	1.0	17.915	17.066	15.660	14.925	14.767	14.725	14.715
	1.5	14.967	14.085	13.173	13.340	13.049	12.047	11.492
C	2.0	13.757	11.788	10.588	11.815	12.545	12.010	11.569
C C	2.5	11.819	10.048	9.717	12.054	13.794	13.992	13.837
C	3.0	11.165	9.824	10.164	13.339	16.095	17.235	17.480
	3.5	11.519	10.546	11.380	15.288	19.163	21.453	22.195
	4.0	12.566	11.875	13.101	17.733	22.879	26.522	27.851
	0.5	15.350	15.153	14.344	12.775	10.990	9.760	---
	1.0	9.866	10.351	11.750	12.135	10.839	10.195	---
	1.5	10.126	10.763	12.257	11.734	11.444	10.256	---
C S S C	2.0	9.822	9.794	9.591	11.099	11.868	11.398	---
	2.5	9.561	8.925	9.606	12.079	13.858	14.247	---
	3.0	9.891	9.451	10.638	13.574	16.768	18.242	---
	3.5	10.562	10.711	12.268	16.249	20.652	23.199	---
	0.5	14.026	13.570	12.185	9.997	7.602	6.247	6.076
	1.0	5.416	5.560	6.113	6.517	6.134	5.565	5.416
S S S S	1.5	4.933	5.105	6.160	7.629	6.432	5.123	4.829
	2.0	5. 416	6.120	6.342	6.934	6.812	5.843	4.932
	2.5	4.854	5.523	4.521	7.102	7.453	6.121	4.903
	3.0	4. 932	5.121	6.481	7.213	6.632	5.932	5.415
	3.5	9.948	5.231	6.392	7.314	6.521	5.521	---

Boundary conditions	Side ratio a/b	ANGLE OF ORTHOTROPICITY						
		0	15	30	45	60	75	90
	0.5	12.807	11.695	9.069	6.447	4.878	4.365	---
	1.0	3.431	3.180	2.679	2.261	1.869	--	---
	1.5	1.698	1.487	1.227	1.339	1.303	0.940	---
S F S S	2.0	1.093	0.768	0.428	0.876	1.090	0.755	---
	2.5	0.814	.321	0.099	0.555	0.573	0.670	---
	3.0	0.662	---	0.165	0.288	0.907	0.624	---
	3.5	0.572	0.206	---	0.075	.855	0.594	---
	0.5	3.122	2.722	1.967	1.307	0.983	0.957	1.000
	1.0	0.780	0.599	0.391	0.280	0.238	0.238	0.250
	1.5	0.347	0.188	0.078	0.086	0.101	0.105	0.110
C F F F	2.0	0.196	0.027	---	0.015	0.054	0.058	0.062
	2.5	0.125	---	---	---	0.032	0.037	0.040
	3.0	0.087	---	---	---	0.020	0.025	0.028
	3.5	0.064	---	.157	---	0.013	0.018	0.020

TABLE 13

Buckling loads ($K_b = N_1 b^2/\pi^2 D'_{22}$) of specially orthotropic plates (Maple Plywood $D'_{11}/D'_{22} = 3.117$, $D'_{12}/D'_{22} = .12$; $D'_{66}/D'_{22} = .26$)

Angle of Orthotropicity Degrees	Side ratio a/b	CCCC		SSCC		SSSS	
		From Equn. 3.34	From Ref. 16	From Equn. 3.34	From Ref. 16	From Eqn. 3.34	From Ref. 16
0	0.5	52.571	52.548	15.350	15.541	14.036	14.026
	1.0	17.915	18.208	9.86	10.181	5.416	5.416
	1.5	14.967	15.429	10.126	10.275	4.933	4.933
	2.0	13.787	12.391	10.822	10.181	5.416	5.416
	2.5	11.815	11.850	9.561	9.925	4.854	4.855
	3.0	11.165	11.943	9.891	10.181	4.932	4.933
	3.5	11.519	11.725	10.562	11.280	4.948	4.949

APPENDIX - D

FIGURES

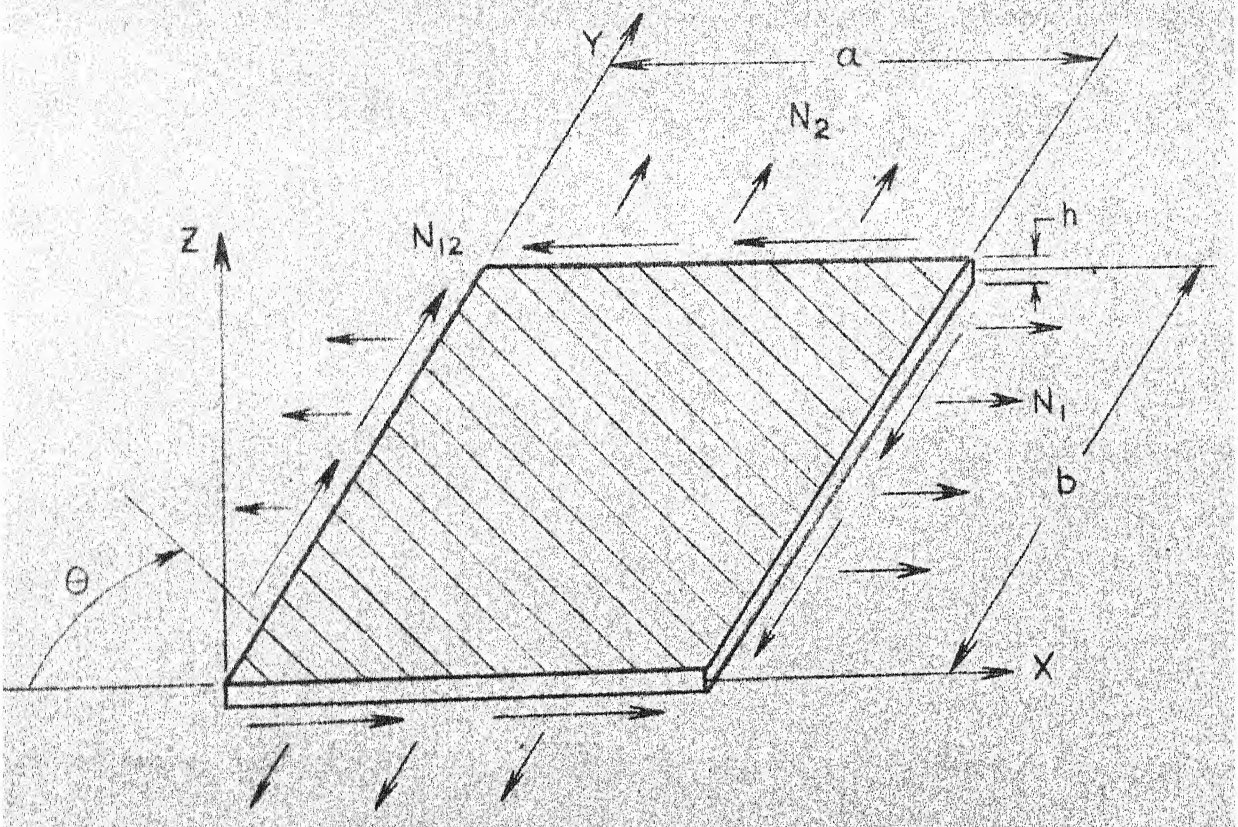

FIG.1 PLATE GEOMETRY

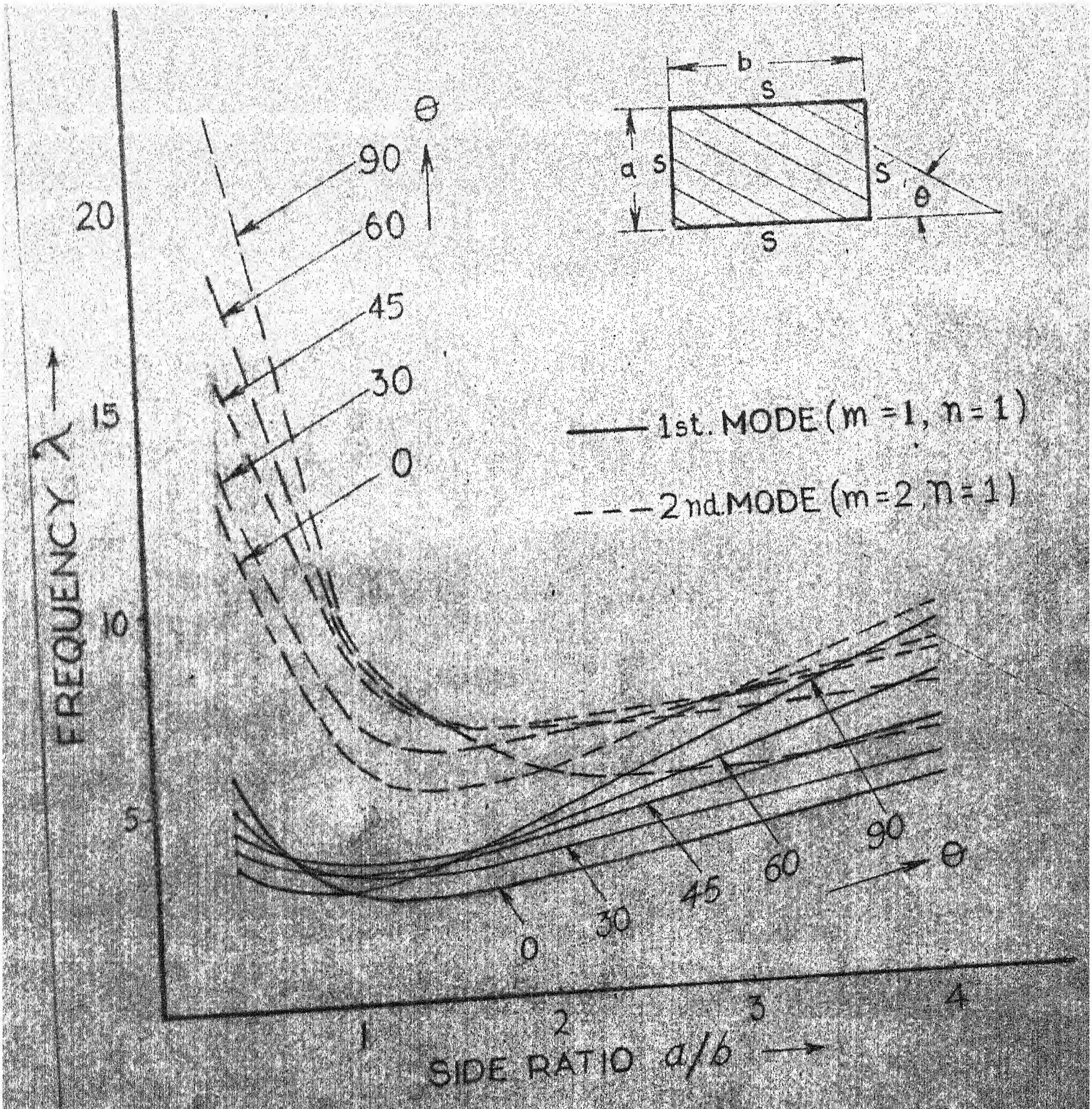

FIG.2. NATURAL FREQUENCY OF GENERALLY ORTHOTROPIC SSSS PLATES

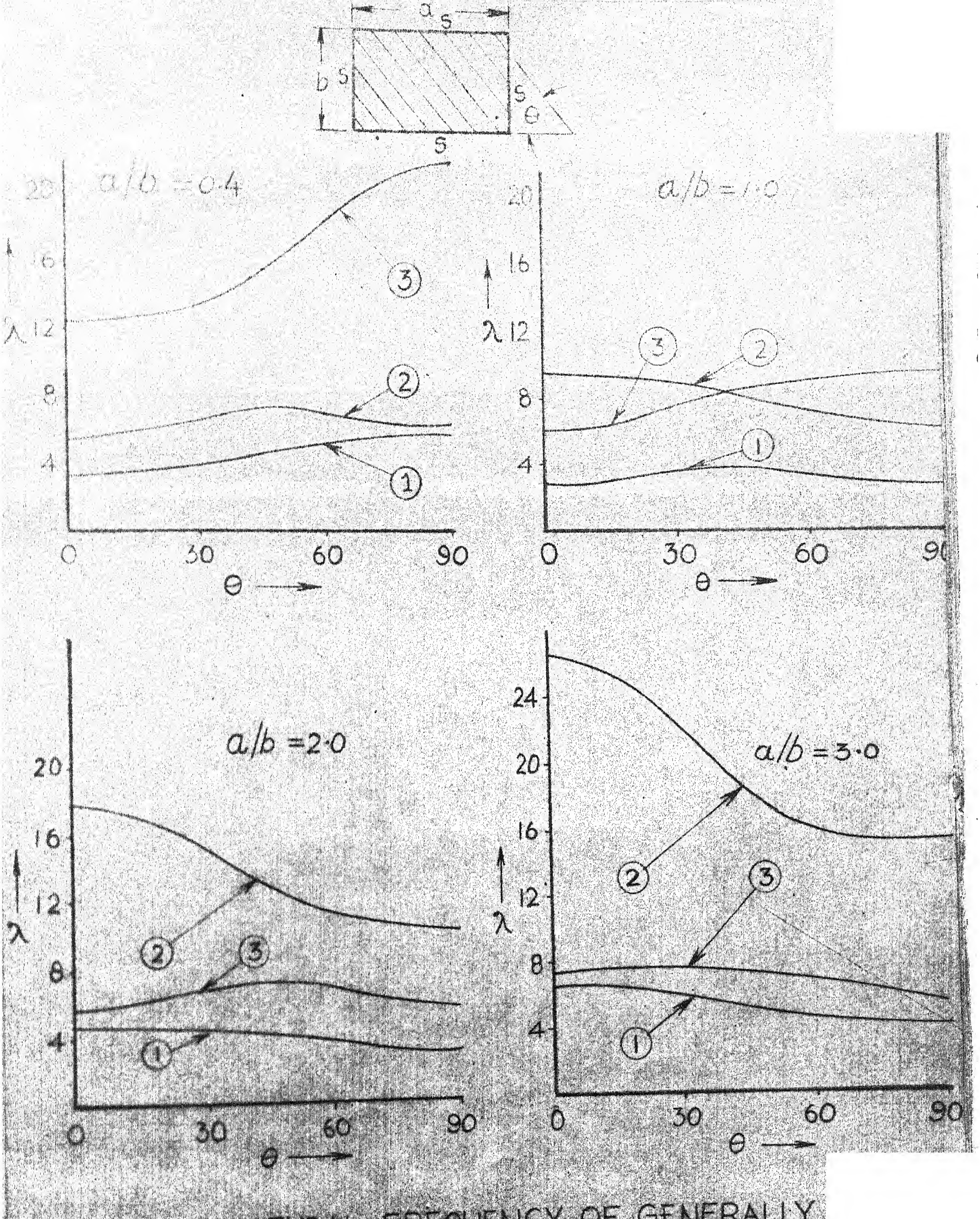

FIG.3 NATURAL FREQUENCY OF GENERALLY ORTHOTROPIC SSSS PLATES

① → m = 1, n = 1 ; ② → m = 1, n = 2 ; ③ → m = 2, n = 1.

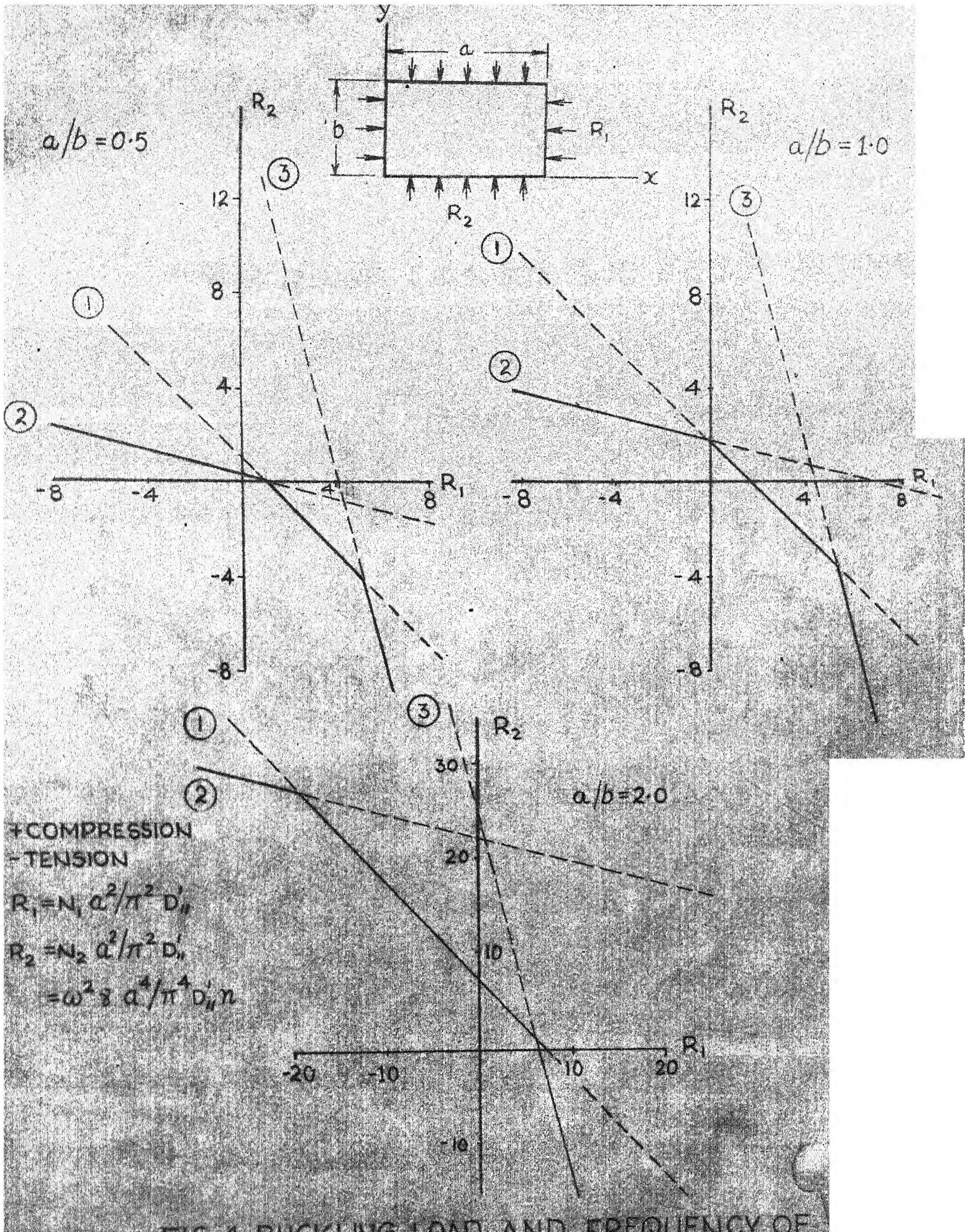

FIG.4. BUCKLING LOAD AND FREQUENCY OF SPECIALLY ORTHOTROPIC SSSS PLATE WITH INPLANE LOAD

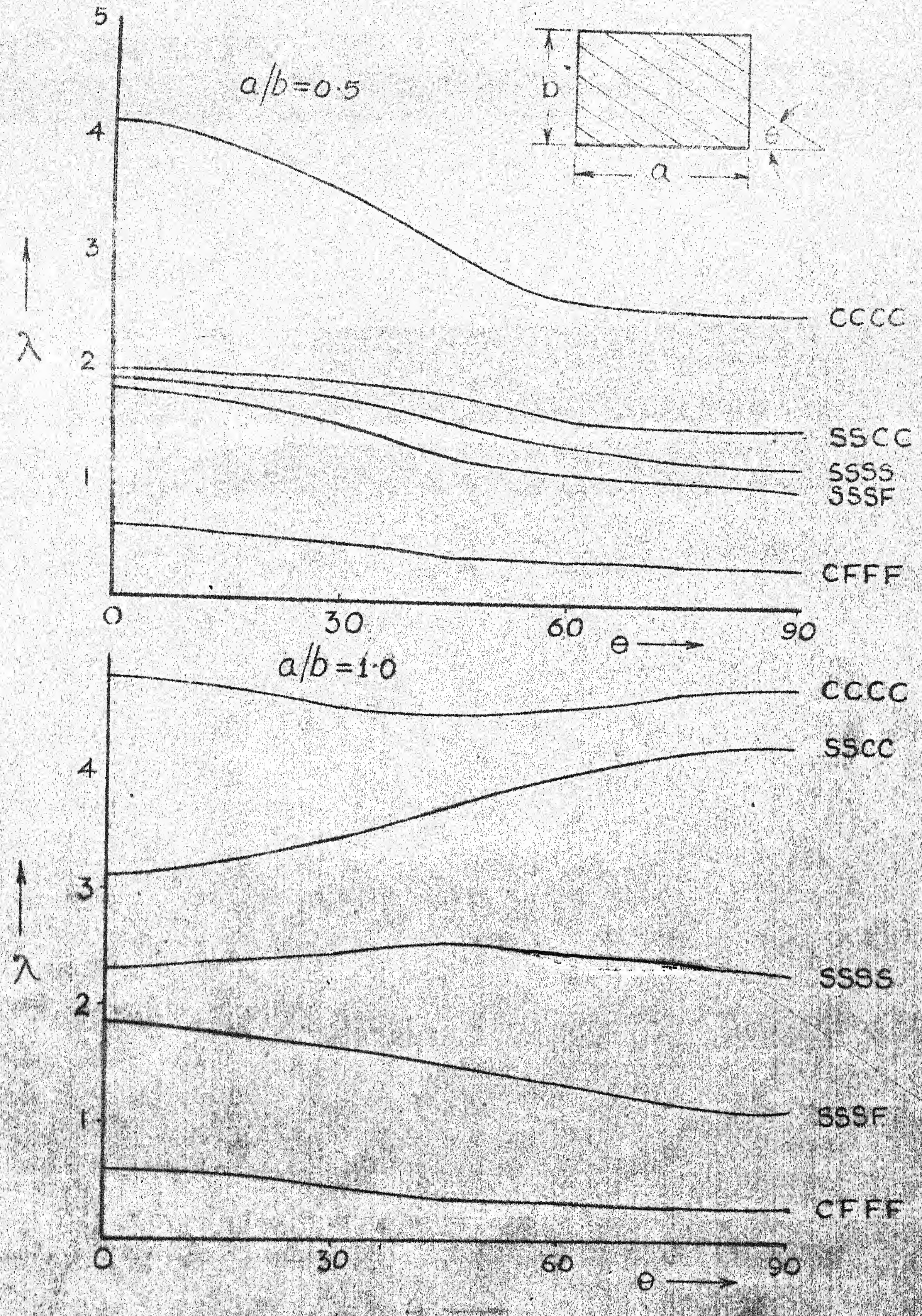

FIG. 5 NATURAL FREQUENCY OF GENERALLY ORTHOTROPIC PLATES WITH VARIOUS B.C.

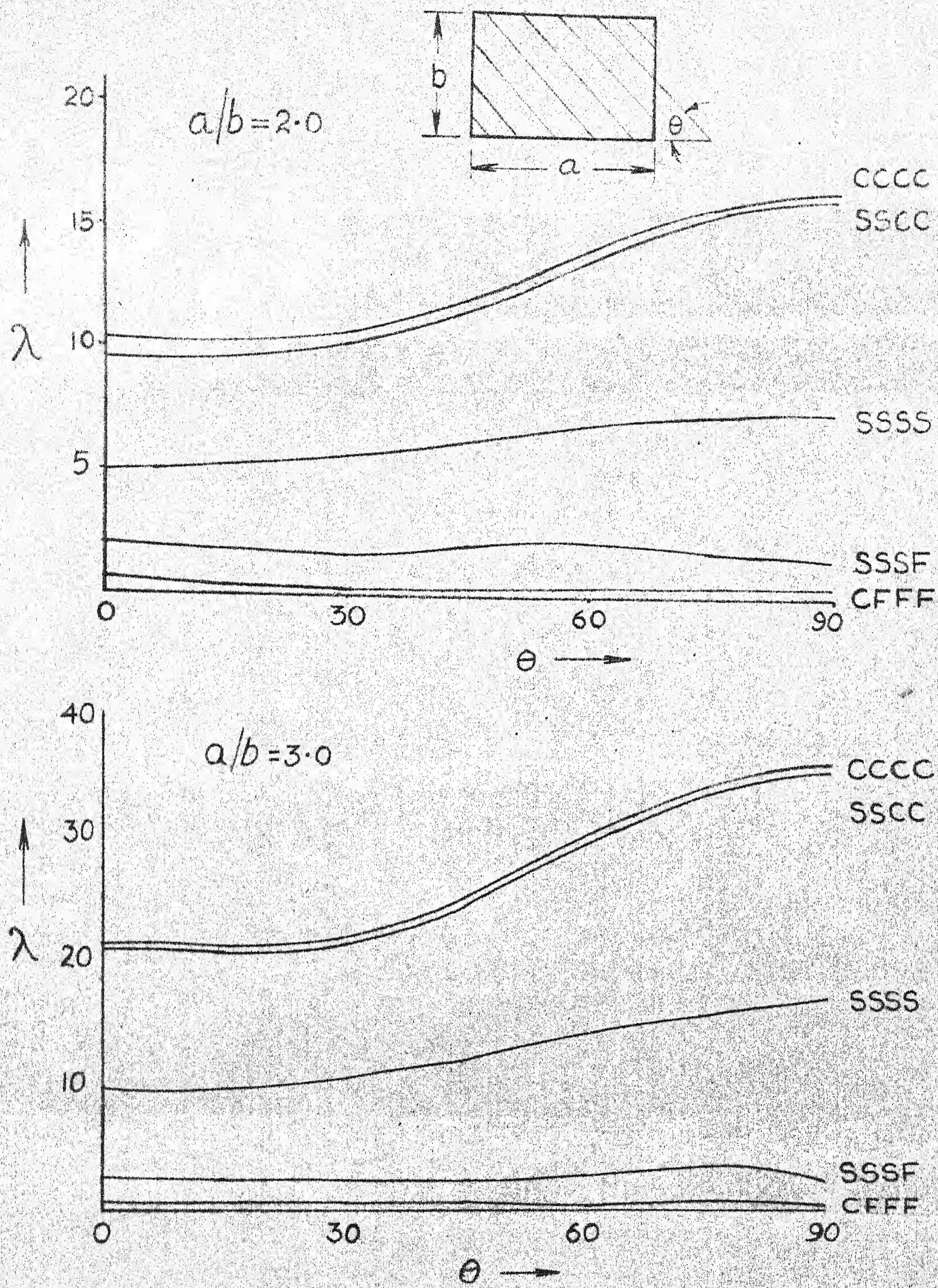

FIG. 6 NATURAL FREQUENCY OF GENERALLY ORTHOTROPIC PLATES WITH VARIOUS B.C.

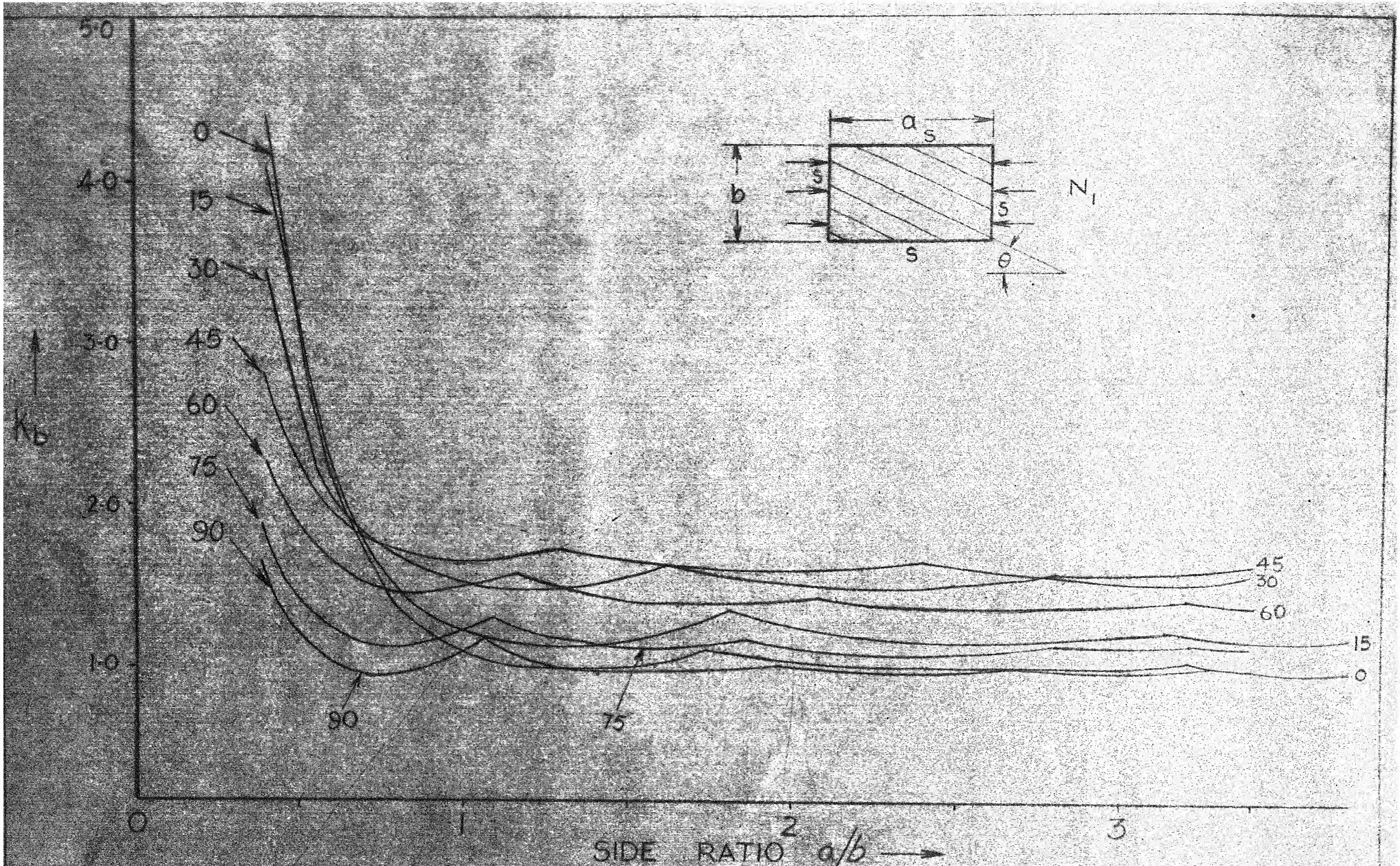

FIG.7 BUCKLING LOADS OF GENERALLY ORTHOTROPIC SSSS PLATE

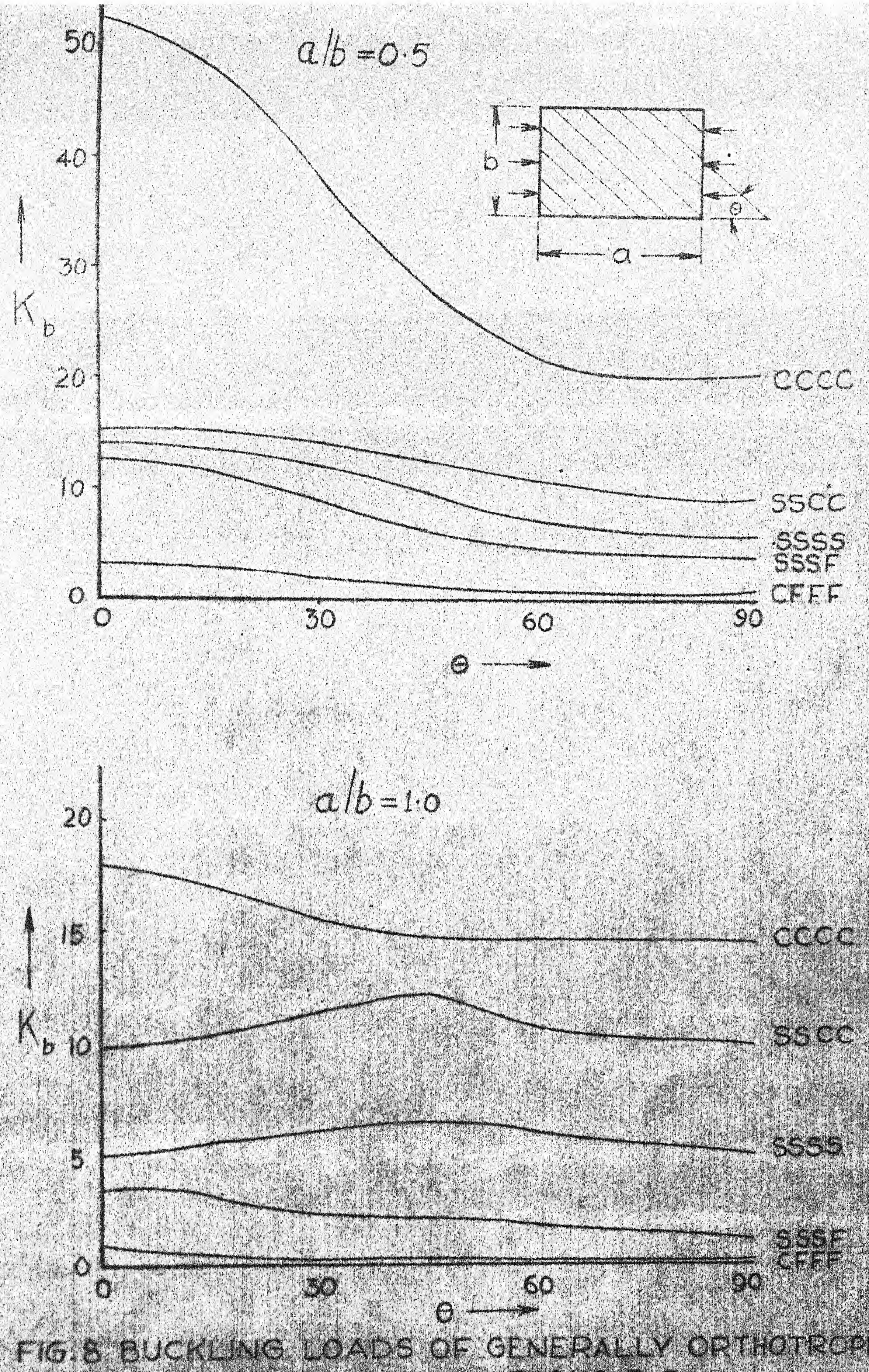

FIG. 8 BUCKLING LOADS OF GENERALLY ORTHOTROPIC PLATES WITH VARIOUS B.C.

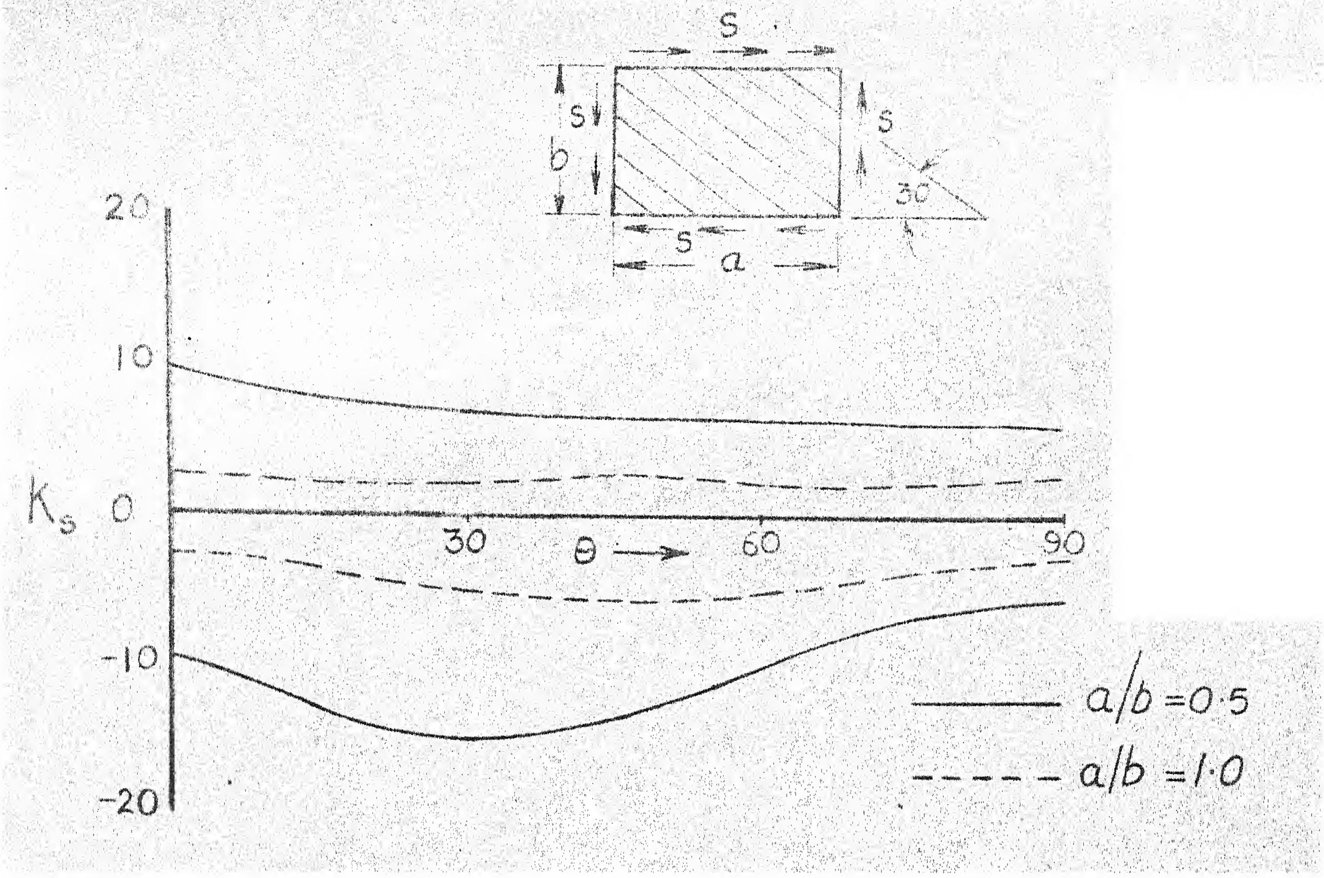

FIG. 9 SHEAR BUCKLING LOADS OF GENERALLY ORTHOTROPIC SSSS PLATES